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A CHARACTERIZATION OF α-CONVOLUTIONS

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For the terminology and notation used here, see [2]. In particular, $\mathfrak{P}$ denotes the class of all probability measures defined on Borel subsets of the positive half-line. By $E_a\,(a \geqslant 0)$ we denote the probability measure concentrated at the point a. For any a $(a > 0)$, the transformation T_a of $\mathfrak{P}$ onto itself is defined by means of the formula $(T_aP)(\mathscr{A}) = P(a^{-1}\mathscr{A})$, where $P \in \mathfrak{P}$, $\mathscr{A}$ is a Borel set, and $a^{-1}\mathscr{A} = \{a^{-1}x\colon x \in \mathscr{A}\}$. The transformation T_0 is defined by assuming $T_0P = E_0$ for all $P \in \mathfrak{P}$.

A commutative and associative $\mathfrak{P}$-valued binary operation $\circ$ defined on $\mathfrak{P}$ is called a *generalized convolution* if it satisfies the following conditions:

(i) $E_0 \circ P = P$ for all $P \in \mathfrak{P}$;

(ii) $(aP + bQ)\circ R = a(P\circ R) + b(Q\circ R)$, whenever $P, Q, R \in \mathfrak{P}$ and $a \geqslant 0, b \geqslant 0, a+b = 1$;

(iii) $(T_aP)\circ(T_aQ) = T_a(P\circ Q)$ for any $P, Q \in \mathfrak{P}$ and $a \geqslant 0$;

(iv) if $P_n \to P$, then $P_n\circ Q \to P\circ Q$ for all $Q \in \mathfrak{P}$, where the convergence is the weak convergence of probability measures;

(v) there exists a sequence $c_1, c_2, \ldots$ of positive numbers such that the sequence $T_{c_n}E_1^{\circ n}$ weakly converges to a measure Q different from E_0 (the power $E_a^{\circ n}$ is taken here in the sense of the operation $\circ$).

The class $\mathfrak{P}$ with a generalized convolution $\circ$ is called a *generalized convolution algebra* and denoted by $(\mathfrak{P}, \circ)$. Algebras admitting a non-trivial homomorphism into the real field are called *regular*.

An algebra $(\mathfrak{P}, \circ)$ is called *quasi-regular* if it satisfies the following condition:

(vi) there exists a sequence $c_1, c_2, \ldots$ of positive numbers such that

$$\lim_{n\to\infty} c_n = 0 \quad \text{and} \quad \lim_{n\to\infty} T_{c_n}E_1^{\circ n} = Q \quad \text{and} \quad Q \neq E_0.$$

It is known that every regular algebra $(\mathfrak{P}, \circ)$ is quasi-regular (see [2], Theorem 4). The following problem appears still to be open:

PROBLEM. *Is every quasi-regular convolution algebra regular?* **(P 826)**

The *a-convolutions*, being a modification of the ordinary convolution, are simple examples of regular generalized convolutions. For every $a > 0$, an a-convolution is defined by the formula

$$\int_0^\infty f(x)(P\circ R)(dx) = \int_0^\infty \int_0^\infty f\big((x^a+y^a)^{1/a}\big)P(dx)R(dy)$$

for all bounded continuous functions f on the positive half-line.

The aim of the present paper is to give a characterization of the a-convolutions. We say that a probability measure P is *infinitely divisible* in the algebra $(\mathfrak{P}, \circ)$ if for every integer n there exists a probability measure P_n such that $P_n^{\circ n} = P$. Further, we say that a probability measure P is *decomposable* if it can be written in the form $P = R_1 \circ R_2$, where $R_1 \neq E_0$ and $R_2 \neq E_0$. It is clear that each infinitely divisible measure is decomposable. Therefore, our result can be regarded as a partial solution of the following problem raised by K. Urbanik:

PROBLEM. *Suppose that $(\mathfrak{P}, \circ)$ is a quasi-regular convolution algebra and the measure E_1 is decomposable. Is then $(\mathfrak{P}, \circ)$ an a-convolution algebra?* **(P 827)**

THEOREM. *Let $(\mathfrak{P}, \circ)$ be a quasi-regular convolution algebra in which the measure E_1 is infinitely divisible. Then $(\mathfrak{P}, \circ)$ is an a-convolution algebra.*

Before proving the Theorem we shall prove some lemmas.

LEMMA 1. *If an algebra $(\mathfrak{P}, \circ)$ is quasi-regular and there exists a sequence $a_1, a_2, \ldots$ such that $T_{a_n} E_1^{\circ n} \to P$, where $P \in \mathfrak{P}$ and $P \neq E_0$, then*

$$\lim_{n\to\infty} a_n = 0.$$

Proof. If the algebra $(\mathfrak{P}, \circ)$ is quasi-regular, then there exists a sequence $c_1, c_2, \ldots$ of positive numbers for which

$$\lim_{n\to\infty} c_n = 0 \quad \text{and} \quad \lim_{n\to\infty} T_{c_n} E_1^{\circ n} = Q, \quad \text{where } Q \in \mathfrak{P} \text{ and } Q \neq E_0.$$

Now, let us suppose that there exists a subsequence $a_{k_1}, a_{k_2}, \ldots$ of the sequence $a_1, a_2, \ldots$ such that

$$\lim_{n\to\infty} a_{k_n}^{-1} = a < \infty.$$

We have

$$T_{a_{k_n}^{-1}} T_{a_{k_n}} T_{c_{k_n}} E_1^{\circ k_n} = T_{c_{k_n}} E_1^{\circ k_n} \to Q.$$

On the other hand,

$$T_{a_{k_n}^{-1}} T_{a_{k_n}} T_{c_{k_n}} E_1^{\circ k_n} = T_{a_{k_n}^{-1}} T_{c_{k_n}} T_{a_{k_n}} E_1^{\circ k_n} \to T_0 P = E_0.$$

Hence, $Q = E_0$, which contradicts the hypothesis.

LEMMA 2. *If for some integer k there exists a probability measure P such that $P^{\circ k} = E_1$, then there exists a point a for which $E_a^{\circ k} = E_1$.*

Proof. It is easy to verify that for some point a

$$\mathrm{P}\left(\left\{x\colon a-\frac{1}{n} \leqslant x \leqslant a+\frac{1}{n}\right\}\right) > 0 \quad \text{for any integer } n > 0.$$

Let us introduce the notation

$$\mathscr{I}_n = \left\{x\colon a-\frac{1}{n} \leqslant x \leqslant a+\frac{1}{n}\right\}$$

and

$$P_n(\mathscr{A}) = \frac{\mathrm{P}(\mathscr{I}_n \cap \mathscr{A})}{\mathrm{P}(\mathscr{I}_n)} \quad \text{for all Borel sets } \mathscr{A}.$$

Then we have $\mathrm{P} = \alpha_n P_n + \beta_n R_n$, where R_n is a probability measure concentrated on the set $[0, \infty) - \mathscr{I}_n$ and $\alpha_n > 0$. Taking into account the formula

$$E_1 = \alpha_n^k P_n^{\circ k} + \sum_{r=1}^{n} \binom{n}{r} \alpha_n^r \beta_n^{n-r} P_n^{\circ r} \circ Q_n^{\circ(n-r)}$$

and the inequality $\alpha_n^k > 0$, we infer that the measure $P_n^{\circ k}$ is concentrated at the point 1. Consequently, $P_n^{\circ k} = E_1$. Further, it is easy to verify that

$$\lim_{n\to\infty} P_n^{\circ k} = E_a^{\circ k}.$$

Thus $E_a^{\circ k} = E_1$, which completes the proof.

Proof of the Theorem. From lemmas 1 and 2 it follows that for the measure E_1 there exists a sequence $a_1, a_2, \ldots$ such that

(1) $$E_1 = T_{a_n} E_1^{\circ n}$$

and

(2) . $$\lim_{n\to\infty} a_n = 0.$$

First of all, we prove that

(3) $$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = 1.$$

Of course, it suffices to prove that for every convergent subsequence a_{n_k}/a_{n_k+1} of sequence a_n/a_{n+1}

$$\lim_{k\to\infty} \frac{a_{n_k}}{a_{n_k+1}} = 1.$$

We prove that

$$\lim_{k\to\infty} \frac{a_{n_k}}{a_{n_k+1}} < \infty. \tag{4}$$

Contrary to this let us suppose that

$$\lim_{k\to\infty} \frac{a_{n_k+1}}{a_{n_k}} = 0.$$

Of course, we have the formula

$$T_{a_{n_k+1}} E_1^{\circ(n_k+1)} = E_1.$$

On the other hand, putting $d_k = a_{n_k+1}/a_{n_k}$,

$$T_{a_{n_k+1}} E_1^{\circ(n_k+1)} = T_{d_k a_{n_k}} E_1^{\circ(n_k+1)} = T_{d_k a_{n_k}} E_1^{\circ n_k} \circ T_{d_k a_{n_k}} E_1 \to E_0.$$

Hence $E_0 = E_1$ which gives a contradiction. Formula (4) is thus proved.

Let $r_k = a_{n_k}/a_{n_k+1}$ and $r_k \to r$. Then we have the relation

$$T_{a_{n_k}} E_1^{\circ(n_k+1)} = T_{a_{n_k}} E_1^{\circ n_k} \circ T_{a_{n_k}} E_1 \to E_1 \circ E_0 = E_1.$$

On the other hand,

$$T_{a_{n_k}} E_1^{\circ(n_k+1)} = T_{r_k a_{n_k+1}} E_1^{\circ(n_k+1)} = T_{r_k} E_1 \to T_r E_1.$$

Hence $E_1 = E_r$, and $r = 1$. Formula (3) is thus proved.

From (2) and (3) it follows that for any pair x, y of positive numbers there exist subsequences $a_{n_1}, a_{n_1}, \ldots$ and $a_{m_1}, a_{m_2}, \ldots$ of the sequence $a_1, a_2, \ldots$ such that

$$\lim_{k\to\infty} \frac{a_{n_k}}{a_{m_k}} = \frac{y}{x}.$$

Moreover, we can assume without loss of generality that the limit

$$s = \lim_{k\to\infty} \frac{a_{n_k}}{a_{m_k+n_k}},$$

perhaps infinite, does exist. First of all, we prove that the limit s is finite. Let us suppose to the contrary that

$$\lim_{k\to\infty} v_k = 0, \quad \text{where } v_k = \frac{a_{n_k+m_k}}{a_{n_k}}.$$

Setting $w_k = a_{n_k}/a_{m_k}$, we have

$$T_{a_{n_k+m_k}} E_1^{\circ(n_k+m_k)} = E_1.$$

On the other hand,

$$T_{a_{n_k+m_k}} E_1^{\circ(n_k+m_k)} = T_{v_k a_{n_k}} E_1^{\circ n_k} \circ T_{v_k w_k a_{m_k}} E_1^{\circ m_k}$$
$$= T_{v_k} E_1 \circ T_{v_k w_k} E_1 \to E_0 \circ E_0 = E_0.$$

Hence $E_0 = E_1$, which is impossible. The finiteness of the limit s is thus proved.

Using the notations $s_k = a_{n_k}/a_{n_k+m_k}$ and $w_k = a_{n_k}/a_{m_k}$, we obtain the following equations:

$$T_{xa_{n_k}} E_1^{\circ(n_k+m_k)} = T_{xa_{n_k}} E_1^{\circ n_k} \circ T_{xa_{n_k}} E_1^{\circ m_k}$$
$$= T_x E_1 \circ T_{xw_k a_{m_k}} E_1^{\circ m_k} \to T_x E_1 \circ T_y E_1 = E_x \circ E_y,$$
$$T_{xa_{n_k}} E_1^{\circ(n_k+m_k)} = T_{xs_k a_{n_k+m_k}} E_1^{\circ(n_k+m_k)} \to T_{xs} E_1 = E_{xs}.$$

Hence we have

$$(5) \qquad E_x \circ E_y = E_{sx}.$$

We define an auxiliary function $g(x, y)$ by means of the formulas $g(x, 0) = x$, $g(0, y) = y$ and $g(x, y) = sx$ for $x > 0$, $y > 0$. The function g satisfies the equation

$$(6) \qquad E_x \circ E_y = E_{g(x,y)}.$$

It is easy to see that g is the only function satisfying (6).

As a direct consequence of equation (6) and of the uniqueness of its solution, we obtain

$$(7) \qquad g(x, y) = g(y, x),$$

$$(8) \qquad g\big(g(x, y), z\big) = g\big(x, g(y, z)\big),$$

$$(9) \qquad g(zx, zy) = zg(x, y)$$

for all non-negative numbers x, y and z.

Now, we prove that the function g is continuous in the quadrant $x \geqslant 0$, $y \geqslant 0$. Let $x_n \to x$ and $y_n \to y$. Moreover, suppose that the sequence $g(x_n, y_n) \to z$, where $0 \leqslant z \leqslant \infty$. The equation $z = \infty$ is impossible. Indeed, setting $p_n = x_n/g(x_n, y_n)$ and $q_n = y_n/g(x_n, y_n)$ by (6), we have $E_{p_n} \circ E_{q_n} = E_1$. If $p_n \to 0$ and $q_n \to 0$, then $E_{p_n} \circ E_{q_n} \to E_0$ and $E_1 = E_0$. It is impossible. Hence

$$z < \infty \quad \text{and} \quad E_x \circ E_y = \lim_{n\to\infty} (E_{x_n} \circ E_{y_n}) = \lim_{n\to\infty} E_{z_n} = E_z.$$

From this equation it follows that $g(x, y) = z$. Thus the function g is continuous.

From (6) we obtain

$$(10) \qquad E_1^{\circ 2} = E_{g(1,1)}.$$

If $g(1,1) < 1$, then, by induction from (10), we get $E_1^{\circ 2^n} = E_{g^n(1,1)} \to E_0$. Hence $T_{a_{2^n}} E_1^{\circ 2^n} \to E_0$. On the other hand, $T_{a_{2^n}} E_1^{\circ 2^n} = E_1$. Since the equation $E_0 = E_1$ cannot hold true, we have the inequality $g(1,1) \geqslant 1$. If $g(1,1) = 1$, then from (10) we get $E_1^{\circ 2} = E_1$ and, consequently, $E_1^{\circ n} = E_1$. Hence, it follows that

$$\lim_{n\to\infty} T_{a_n} E_1^{\circ n} = \lim_{n\to\infty} T_{a_n} E_1 = E_0.$$

Of course, it is impossible. Therefore,

$$g(1,1) > 1. \tag{11}$$

By (9), to prove the inequality

$$g(x,y) > x \qquad (x \geqslant 0, y > 0) \tag{12}$$

it suffices to prove it for $y = 1$. Let us suppose that there exists a number x_1 such that $g(x_1, 1) < x_1$. Since $g(0,1) = 1$ and the function g is continuous, we infer that there exists a number x_0 lying between 0 and x_1 for which the equation $g(x_0, 1) = x_0$ holds. From this equation, (8) and (9) we obtain, by induction, $g\big(x_0, g^n(1,1)\big) = x_0$. Setting $z_n = x_0/g^n(1,1)$, we get, by (9), $g(z_n, 1) = z_n$. From inequality (11) it follows that

$$\lim_{n\to\infty} z_n = 0.$$

Thus, by the continuity of g, the last equation implies $g(0,1) = 0$ which contradicts the definition of $g(0,1) = 1$. This completes the proof of (12).

Now, we prove that for all $x \geqslant 0$

$$g(x, y_1) > g(x, y_2), \qquad \text{whenever } y_1 > y_2. \tag{13}$$

If $y_2 = 0$, then (13) is a consequence of (12) and the definition of g. Suppose that $y_2 > 0$. Since $g(0, y_2) = y_2$ and, by (12), $g(y_1, y_2) > y_1$, we infer, by virtue of continuity of g, that there exists a number y satisfying the inequality $0 < y < y_1$ for which the equation $g(y, y_2) = y_1$ holds. Hence, taking into account (7), (8) and (12), we obtain

$$\begin{aligned} g(x, y_1) &= g\big(x, g(y, y_2)\big) = g\big(g(x,y), y_2\big) \\ &= g\big(g(y,x), y_2\big) = g\big(y, g(x, y_2)\big) = g\big(g(x,y_2), y\big) > g(x, y_2) \end{aligned}$$

which completes the proof of (13).

Bohnenblust proved (see [1], p. 630-632) that the continuous functions g satisfying conditions (7)-(9), (13) and condition $g(0,x) = x$ are of the form

$$g(x,y) = (x^a + y^a)^{1/a}, \qquad \text{where } 0 < a < \infty.$$

From this and from (6) it follows that for every $x \geqslant 0, y \geqslant 0$ the equation

$$E_x \circ E_y = E_{(x^a+y^a)^{1/a}},$$

where a is a positive constant, holds. Now, it is easy to verify that for convex linear combinations P and R of the measures E_a $(a \geqslant 0)$ the formula

$$\int_0^\infty f(x)(P\circ R)(dx) = \int_0^\infty \int_0^\infty f\big((x^a+y^a)^{1/a}\big)P(dx)\,R(dy), \tag{14}$$

where f is a bounded, continuous function on $[0, \infty)$, holds. Since the convex linear combinations of the measures E_a form a dense subset of $\mathfrak{P}$ in the sense of weak convergence, formula (14) holds for all measures P and R from $\mathfrak{P}$. In other words, the algebra in question is an a-convolution algebra.

REFERENCES

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