## SOME REMARKS ON VARIANTS OF THE NAVIER-STOKES EQUATIONS

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In references [4] and [5], Ladyzhenskaya has offered mathematical and physical grounds for the consideration of new equations for the description of the motion of viscous incompressible fluids. She has indicated that such motion may be described by one of the following systems:

(1)
$$\begin{cases} \dfrac{\partial u_i}{\partial t} - \left(\nu_0 + \nu_1 \displaystyle\int_\Omega \left(\dfrac{\partial u_k}{\partial x_l}\right)^2\right) dx\, \dfrac{\partial^2 u_i}{\partial x_j^2} + u_j \dfrac{\partial u_i}{\partial x_j} = -\dfrac{\partial p}{\partial x_i} + f_i \\[4mm] \hspace{4cm} (i = 1, 2, 3), \\[3mm] \dfrac{\partial u_i}{\partial x_i} = 0; \end{cases}$$

(2)
$$\begin{cases} \dfrac{\partial u_i}{\partial t} - \dfrac{\partial}{\partial x_j}\left\{\left(\nu_0 + \nu_1\left(\dfrac{\partial u_k}{\partial x_l}\right)^2\right) \dfrac{\partial u_i}{\partial x_j}\right\} + u_j \dfrac{\partial u_i}{\partial x_j} = -\dfrac{\partial p}{\partial x_i} + f_i \\[4mm] \hspace{4cm} (i = 1, 2, 3), \\[3mm] \dfrac{\partial u_i}{\partial x_i} = 0; \end{cases}$$

or

(3)
$$\begin{cases} \dfrac{\partial u_i}{\partial t} + \left[\operatorname{curl}\left\{\left(\nu_0 + \nu_1(\operatorname{curl} u)^2\right)\operatorname{curl} u\right\}\right]_i + u_j\dfrac{\partial u_i}{\partial x_j} = -\dfrac{\partial p}{\partial x_i} + f_i \\[4mm] \hspace{4cm} (i = 1, 2, 3), \\[3mm] \dfrac{\partial u_i}{\partial x_i} = 0; \end{cases}$$

where $\nu_0$ and $\nu_1$ are positive constants which characterise the medium, $\Omega$ is the domain in $E^3$ containing the fluid, the $x_j$ $(j = 1, 2, 3)$ are rectangular Cartesian coordinates, $t$ denotes the time, and $u$, $p$ and $f$, each functions of $x$ and $t$, represent respectively the velocity, pressure and the external force. Independently of Ladyzhenskaya, physical arguments for the replacement of the Navier-Stokes equations by those of system (3) have been put forward by Golovkin [3].

Ladyzhenskaya [4] maintains that "which of these (or similar) systems will be the most appropriate for the description of viscous incompressible flow will be shown by future comprehensive mathematical and physical analysis. For each of the systems proposed unique solvability in the large of the initial boundary value problems has been established, and this contrasts with the situation in respect of the Navier-Stokes equations. Such a property is of course required for a proper mathematical description of a deterministic physical process."

In view of Ladyzhenskaya's remarks it seems of interest to consider the extent to which solutions of the systems (1), (2) and (3) enjoy properties of the same general character as those possessed by the Navier-Stokes equations. In this note we consider recent studies [1], [6] of lower bounds for solutions of the Navier-Stokes equations and obtain comparable results for each of the systems (1) to (3). By contrast, however, whilst such studies of lower bounds in the Navier-Stokes case readily give rise to well established backward uniqueness theorems, we have not been able to establish that they do so for the newly proposed systems.

The methods of proof of our results for each of systems (1), (2) and (3) are patterned after those in [6], and are so similar that we give detail only in respect of system (1). Results of the same general character can be obtained under rather different hypotheses by adhering more closely to [1]. From this point onward we shall assume that in each of the systems (1), (2) and (3), $f_i = 0$ $(i = 1, 2, 3)$. This is appropriate if the external force may be absorbed into the pressure term.

Suppose $\Omega$ is a bounded domain in $E^3$, with boundary $\partial\Omega$ and closure $\overline{\Omega}$. Suppose, moreover, that $\Omega$ is such that the use of Green's theorem in what follows is justified. We adopt the following notation:

If $g = (g_i)$, $h = (h_i)$ are 3-vector functions belonging to $C^1(\overline{\Omega} \times [0, T))$, where $0 < T \leqslant \infty$, put

$$\left(g(t), h(t)\right) = \int_\Omega g_i(x, t)h_i(x, t)\, dx, \quad |g(t)|^2 = \left(g(t), g(t)\right)$$

and

$$\|g(t)\|^2 = \int_\Omega \frac{\partial}{\partial x_j} g_i(x, t)\frac{\partial}{\partial x_j} g_i(x, t)\, dx.$$

The theorem to follow embodies our main result.

THEOREM 1. *Let $\Omega$ be a bounded domain of $E^3$ and let $u \in C^2(\overline{\Omega} \times [0, T))$, $p \in C^1(\overline{\Omega} \times [0, T))$ be a solution of one of the systems (1), (2) or (3), such that $u = 0$ on $\partial\Omega \times [0, T)$. Suppose $U(t) = \sup\{|u(x, t)| : x \in \Omega\} \in L^2[0, T)$. Then*

(i) $|u(t)| > 0$ *for* $0 \leqslant t_0 \leqslant t < T$ *implies that*

(4)                    $|u(t)| \geqslant K\, |u(t_0)|\, \exp\left(-\lambda(t - t_0)\right), \quad t_0 \leqslant t < T,$

*where $K$ and $\lambda$ are positive constants depending on $u(t_0)$ and the $L^2$ norm of $U$;*

(ii) *if $T = \infty$, then $|u(t)| = O\left(\exp(-\mu t)\right)$ as $t \to \infty$ for each $\mu > 0$ implies that $u \equiv 0$.*

Proof (appropriate to system (1)). With the notational conventions introduced above the system of equations with which we are presently concerned is

$$(5) \qquad \frac{\partial u_i}{\partial t} - (\nu_0 + \nu_1 \|u\|^2)\, \Delta u_i + u_j \frac{\partial u_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} \qquad (i = 1, 2, 3),$$

$$(6) \qquad \frac{\partial u_i}{\partial x_i} = 0.$$

Multiplication of (5) by $u_i$, summation over $i$ and integration over $\Omega$ gives

$$(7) \qquad \frac{1}{2}\frac{d}{dt}\, |u|^2 + (\nu_0 + \nu_1 \|u\|^2)\, \|u\|^2 = 0$$

since $(u, u \cdot \mathrm{grad}\, u) = 0$ and $(u, \mathrm{grad}\, p) = 0$.

Similarly, multiplication of (5) by $\partial u_i/\partial t$ and integration over $\Omega$ provides

$$(u_t, u_t) + \frac{1}{2}\, (\nu_0 + \nu_1 \|u\|^2)\, \frac{d}{dt}\, \|u\|^2 + (u_t, u \cdot \mathrm{grad}\, u) = 0,$$

and thus

$$\frac{d}{dt}\left\{\nu_0 \|u\|^2 + \frac{\nu_1}{2}\, \|u\|^4\right\} = -2\,(u_t, u_t + u \cdot \mathrm{grad}\, u).$$

Let

$$Q(t) = \left\{\nu_0 \|u\|^2 + \frac{\nu_1}{2}\, \|u\|^4\right\}\Big/ |u|^2$$

for $t_0 \leqslant t < T$; plainly,

$$|u|^4 \frac{d}{dt}\, Q(t) = |u|^2 \frac{d}{dt}\left\{\nu_0\|u\|^2 + \frac{\nu_1}{2}\, \|u\|^4\right\} - 2\left\{\nu_0\|u\|^2 + \frac{\nu_1}{2}\cdot\|u\|^4\right\}(u, u_t).$$

From (7) we see that $-(u, u_t) \geqslant 0$, and hence

$$|u|^4 \frac{d}{dt}\, Q(t) \leqslant |u|^2 \frac{d}{dt}\left\{\nu_0\|u\|^2 + \frac{\nu_1}{2}\, \|u\|^4\right\} - 2\,(u, u_t)\{\nu_0\|u\|^2 + \nu_1\|u\|^4\}$$

$$= -2\{|u|^2(u_t, u_t + u \cdot \mathrm{grad}\, u) - (u, u_t)(u, u_t + u \cdot \mathrm{grad}\, u)\}$$

$$= -2\{|u|^2\, |u_t + \tfrac{1}{2} u \cdot \mathrm{grad}\, u|^2 - \tfrac{1}{4}|u|^2\,|u \cdot \mathrm{grad}\, u|^2 -$$

$$-\,(u, u_t + \tfrac{1}{2} u \cdot \mathrm{grad}\, u)^2 + \tfrac{1}{4}(u, u \cdot \mathrm{grad}\, u)^2\}$$

$$\leqslant \tfrac{1}{2}|u|^2\,|u \cdot \mathrm{grad}\, u|^2.$$

Thus,

$$\frac{dQ}{dt} \leqslant \nu_0^{-1} U^2 \{\nu_0 \|u\|^2 / |u|^2\} \leqslant \nu_0^{-1} U^2 Q,$$

and integration of this last inequality shows that

$$Q(t) \leqslant Q(t_0) \exp \int_{t_0}^{t} \nu_0^{-1} U^2(s)\,ds, \quad t_0 \leqslant t < T.$$

Now,

$$\frac{d}{dt} \log |u| = \tfrac{1}{2} |u|^{-2} \frac{d}{dt} |u|^2$$

$$= -(\nu_0 + \nu_1 \|u\|^2) \|u\|^2 |u|^{-2}$$

$$\geqslant -2Q.$$

Therefore,

$$\frac{d}{dt} \log |u| \geqslant -2Q(t_0) \exp \int_{t_0}^{t} \nu_0^{-1} U^2(s)\,ds, \quad t_0 \leqslant t < T,$$

and hence

$$|u(t)| \geqslant |u(t_0)| \exp \left\{ -2Q(t_0) \int_{t_0}^{t} \exp \left( \int_{t_0}^{\eta} U^2(s)\nu_0^{-1}\,ds \right) d\eta \right\}.$$

Part (i) of the theorem is now evident since by hypothesis $U \epsilon L^2[0, T)$. As for part (ii), the arguments of [1] and [6] are immediately applicable. We repeat them here for the sake of completeness. Suppose the conclusion of (ii) is false; then there exists $t_0 \epsilon [0, \infty)$ such that $|u(t_0)| > 0$, and from continuity $|u(t)| > 0$ on some interval $[t_0, t_1)$. Let $[t_0, T)$ be the largest such interval on which $|u(t)| > 0$. This interval must be $[t_0, \infty)$, for if $T$ were finite (4) would imply that $|u(T)| > 0$ giving a contradiction. But then from (4),

$$e^{\lambda t} |u(t)| \geqslant C |u(t_0)| \quad \text{for all } t \epsilon [t_0, \infty),$$

where $C$ is some positive constant, and for this to be compatible with the hypothesis of (ii), $|u(t_0)| = 0$ providing a contradiction.

Note. In following through the proof of Theorem 1 in respect of system (2) it is appropriate to set

$$Q(t) = \left\{ \nu_0 \|u\|^2 + \frac{\nu_1}{2} \int_{\Omega} \left( \frac{\partial u_i}{\partial x_j} \right)^2 \left( \frac{\partial u_k}{\partial x_l} \right)^2 dx \right\},$$

and in respect of system (3) to define

$$Q(t) = \left\{ \nu_0 \|u\|^2 + \frac{\nu_1}{2} \int_{\Omega} |\text{curl}\, u|^4\, dx \right\}.$$

By means of (7) and Poincaré's inequality, we may derive companion to Theorem 1:

**THEOREM 2.** *If $\Omega$ is a bounded domain of $E^3$ and $u \epsilon C^2(\overline{\Omega} \times [0, T))$, $p \epsilon C^1(\overline{\Omega} \times [0, T))$ is a solution of one of the systems* (1), (2) *or* (3) *such that $u \equiv 0$ on $\partial\Omega \times [0, T)$, then*

$$|u(t)|^2 \leqslant |u(0)|^2 e^{-2a\nu_0 t}, \qquad 0 \leqslant t < T,$$

*where $a$ is a positive constant depending on the diameter of $\Omega$.*

The methods used to derive results comparable with Theorems 1 and 2 for the Navier-Stokes system of equations are easily adapted to provide forward and backward uniqueness theorems for solutions of that system. Whilst we have been able to obtain a uniqueness theorem for solutions of systems (1), (2) or (3) forward in time, Theorem 3 below, a proof of backward uniqueness has so far eluded us **(P 713)**.

**THEOREM 3.** *Let $\Omega$ be a bounded domain of $E^3$. Let $u^{(i)} \epsilon C^2(\overline{\Omega} \times [0, T))$, $p^i \epsilon C^1(\overline{\Omega} \times [0, T))$, $i = 1, 2$, be two solutions of one of the systems* (1), (2) *or* (3) *with $u^{(1)} = u^{(2)}$ on $\partial\Omega \times [0, T)$. If $u^{(1)}(x, t_0) = u^{(2)}(x, t_0)$ for all $x \epsilon \Omega$ and some $t_0, 0 \leqslant t_0 < T$, then $u^{(1)} \equiv u^{(2)}$ for all $t \epsilon [t_0, T)$.*

**Proof** (appropriate to system (1)). Let $w_i = u_i^{(1)} - u_i^{(2)}$ ($i = 1, 2, 3$). Using (5), plainly

$$(8) \qquad \frac{\partial w_i}{\partial t} - \nu_0 \Delta w_i - \nu_1 (\|u^{(1)}\|^2 \Delta u_i^{(1)} - \|u^{(2)}\|^2 \Delta u_i^{(2)}) + u_k^{(1)} \frac{\partial}{\partial x_k} w_i + w_k \frac{\partial}{\partial x_k} u_i^{(2)}$$

$$= -\frac{\partial}{\partial x_i} (p^{(1)} - p^{(2)})$$

and

$$(9) \qquad \frac{\partial w_i}{\partial x_i} = 0.$$

Multiplication by $w_i$ of (8), integration over $\Omega$ and appeal to Green's Theorem shows that

$$(10) \qquad \frac{1}{2} \frac{d}{dt} |w|^2 + \nu_0 \|w\|^2 + \int_\Omega w_i w_k \frac{\partial}{\partial x_k} u_i^{(2)} dx +$$

$$+ \frac{\nu_1}{2} \{(\|u^{(1)}\|^2 - \|u^{(2)}\|^2)^2 + (\|u^{(1)}\|^2 + \|u^{(2)}\|^2)\|w\|^2\} = 0.$$

From (10),

$$\frac{1}{2} \frac{d}{dt} |w|^2 \leqslant V(t) |w|^2,$$

where $V(t) = \sup\{|\operatorname{grad} u^{(2)}(x, t)| : x \epsilon \Omega\}$.

Now, if $|w(t_1)| > 0$ for some $t_1 > t_0$, then

$$|w(t_0)| \geqslant |w(t_1)| \exp\left(-\gamma(t_1 - t_0)\right)$$

for some positive constant $\gamma$. But then $|w(t_0)| > 0$, which contradicts the hypothesis.

Remark. Demonstration of the theorem in the cases of solutions of systems (2) and (3) rests on the following respective equalities companion to (10):

$$\frac{1}{2}\frac{d}{dt}|w|^2 + \nu_0\|w\|^2 + \int_\Omega w_i w_k \frac{\partial}{\partial x_k} u_i^{(2)}\, dx +$$

$$+ \frac{\nu_1}{2}\left\{\int_\Omega\left[\left\{\frac{\partial w_i}{\partial x_k}\left(\frac{\partial u_i^{(1)}}{\partial x_k} + \frac{\partial u_i^{(2)}}{\partial x_k}\right)\right\}^2 + \left(\frac{\partial w_i}{\partial x_k}\right)^2\left\{\left(\frac{\partial u_j^{(1)}}{\partial x_i}\right)^2 + \left(\frac{\partial u_j^{(2)}}{\partial x_i}\right)^2\right\}\right]dx\right\} = 0$$

and

$$\frac{1}{2}\frac{d}{dt}|w|^2 + \nu_0\|w\|^2 + \int_\Omega w_i w_k \frac{\partial}{\partial x_k} u_i^{(2)}\, dx +$$

$$+ \frac{\nu_1}{2}\left\{\int_\Omega\left[\{(\mathrm{curl}\, u^{(1)})^2 - (\mathrm{curl}\, u^{(2)})^2\}^2 + \{(\mathrm{curl}\, u^{(1)})^2 + (\mathrm{curl}\, u^{(2)})^2\}\,(\mathrm{curl}\, w)^2\right]dx\right\} = 0.$$

In conclusion we observe that the time independent equations (1) have yet another feature in common with the Navier-Stokes equations in that solutions of them possess the strong unique continuation property. This can be shown by an elementary application of the methods of [2].

## REFERENCES •

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