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ON INTEGRABILITY OF GF-STRUCTURE

BY

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Structures on differentiable manifold by introducing vector-valued linear functions satisfying some algebraic equations have been extensively studied by a number of mathematicians under various topics such as almost complex, complex, almost product, contact and almost contact structures.

Recently, Duggel (1) defined on a differentiable manifold a new structure, called *GF-structure*, which is more general than almost complex, almost product and almost tangent structures. The purpose of the present paper is to study the integrability conditions of *GF*-structure manifold by defining two distributions in the manifold.

1. Introduction. Let us consider a differentiable manifold M_n of class C^∞. If there exists on M_n a vector-valued linear function F of class C^∞ such that

$$\overline{\overline{X}} = a^2 X, \tag{1.1}$$

where $\overline{X} = F(X)$, and a is any complex number, then F is said to give a *differentiable structure*, briefly *GF*-structure, to M_n defined by (1.1). It is well known (1) that M_n is endowed with an almost product structure or an almost complex structure or an almost tangent structure according to as $a = 1, -1$ or $a = i, -i$ or $a = 0$.

Let us define two operators l and m by

$$2al = aI+F \quad \text{and} \quad 2am = aI-F \quad (a \neq 0), \tag{1.2}$$

where I is the unit tensor.

THEOREM 1.1. *The operators given by* (1.2) *are complementary projection operators, that is,*

$$l^2 = l, \quad m^2 = m, \quad l+m = I, \quad lm = ml = 0.$$

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(1) K. L. Duggel, *On differentiable structures defined by algebraic equations, I. Nijenhuis tensor*, Tensor, New Series, 22 (1971), p. 238-242.

Proof. In fact,

$$l^2 = \frac{1}{4a^2}(aI+F)(aI+F) = \frac{1}{4a^2}(2a^2I+2aF) = l.$$

Similarly, we can show that $m^2 = m$. Again,

$$lm = \frac{1}{4a^2}(aI+F)(aI-F) = \frac{1}{4a^2}(a^2I-F^2) = 0.$$

Similarly, $ml = 0$. Adding l and m, we get $l+m = I$, and hence the result follows.

THEOREM 1.2. *The structure F acts on l and m as follows:*

$$lF = Fl = a, \quad Fm = -am = mF. \tag{1.3}$$

Proof. In fact,

$$Fl = \frac{1}{2a}(aF+F^2) = \frac{1}{2a}(aF+a^2I) = \frac{1}{2}(F+aI) = \frac{1}{2}\cdot 2al = al.$$

Similarly, $lF = al$. Again,

$$Fm = \frac{1}{2a}(aF-F^2) = \frac{1}{2a}(aF-a^2I) = \frac{1}{2}(F-aI) = \frac{1}{2}(-2am) = -am.$$

Similarly, $mF = -am$, and hence the result follows.

2. Nijenhuis tensor. Let N be the Nijenhuis tensor of type $(1, 2)$ corresponding to the structure F, given by

$$N(X, Y) = [FX, FY]-F[FX, Y]-F[X, FY]+a^2[X, Y] \tag{2.1}$$

for any two vector fields X and Y.

By virtue of Theorems 1.1 and 1.2 we get

$$l(N(mX, mY)) = 4a^2l[mX, mY], \tag{2.2}$$

$$m(N(lX, lY)) = 4a^2m[lX, lY], \tag{2.3}$$

$$l(N(lX, lY)) = 0, \tag{2.4}$$

$$m(N(mX, mY)) = 0, \tag{2.5}$$

$$N((mX, lY)) = 0, \tag{2.6}$$

$$N((lX, mY)) = 0. \tag{2.7}$$

3. Integrability conditions. Let L and M be the distributions corresponding to the operators l and m. We can easily verify that the distribution M is integrable if and only if

$$l([mX, mY]) = 0. \tag{3.1}$$

THEOREM 3.1. *The distribution M is integrable if and only if $l(N(mX, mY)) = 0$ for any two vector fields X and Y.*

This theorem follows from (3.1) and (2.2).

It is well known that L is integrable if and only if

$$m([lX, lY]) = 0 \tag{3.2}$$

for any two vector fields X and Y. In consequence of (3.2) and (2.3) we have the following theorem:

THEOREM 3.2. *The distribution L is integrable if and only if $m(N(l(X), l(Y))) = 0$ for any two vector fields X and Y.*

Since $l+m = I$, we have

$$\begin{aligned} N(X, Y) = {} & l(N(l(X), l(Y)))+l(N(m(X), l(Y)))+ \\ & +l(N(l(X), m(Y)))+l(N(m(X), m(Y)))+m(N(l(X), l(Y)))+ \\ & +m(N(m(X), l(Y)))+m(N(l(X), m(Y)))+m(N(m(X), m(Y))). \end{aligned} \tag{3.3}$$

In view of (3.3) and (2.4)-(2.7) we have

$$N(X, Y) = l(N(mX, mY))+m(N(lX, lY)) \tag{3.4}$$

for any two vector fields X and Y. In view of Theorems 3.1 and 3.2 and equation (3.4) we have

THEOREM 3.3. *Both distributions L and M are integrable if and only if the Nijenhuis tensor vanishes.*

The *Lie derivative* $£_Y F$ of the tensor field F with respect to the vector field Y is, by definition, a tensor field of the same type as F given by

$$(£_Y F)X = F[X, Y]-[FX, Y].$$

In consequence of (1.3) we have

$$(£_{mY} F)l(X) = F[lX, mY]-[alX, mY] = F[lX, mY]-a[lX, mY],$$

whence we get

$$l(£_{mY} F)l(X) = al[lX, mY]-al[lX, mY] = 0.$$

Similarly, $m(\pounds_{lY}F)mX = 0$. Thus we have

THEOREM 3.4. *The tensor fields* $l(\pounds_{mY}F)l$ *and* $m(\pounds_{lY}F)m$ *vanish identically.*

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