COLLOQUIUM MATHEMATICUM

VOL. XL 1979 FASC. 2

A NOTE ON THE PREDUAL OF Lip(S, d)

BY

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In what follows* Lip(S, d) will denote the Banach space of real-valued functions f on the metric space (S, d) satisfying

$$\|f\| = \max(\|f\|_\infty, \|f\|_d) < \infty,$$

$$\text{where } \|f\|_d = \sup_{s \neq t} \frac{|f(s)-f(t)|}{d(s, t)} \text{ and } \|f\|_\infty = \sup_s |f(s)|.$$

We will use ε_s to denote the point evaluation functional defined by $\varepsilon_s(f) = f(s)$. The closure of the linear span of the point evaluations in the dual space Lip$(S, d)^*$ will be denoted by $\Delta(S, d)$. (We will usually omit d and simply write Lip(S) or $\Delta(S)$.) *Isomorphism* will mean linear homeomorphism and *isometric* will mean linearly isometric. In [2], Corollary 4.2, it is shown that $\Delta(S)^*$ is canonically isometric to Lip(S) and we will refer to $\Delta(S)$ as to the predual of Lip(S). We denote by I the unit interval with the usual metric.

It is known that non-isomorphic Banach spaces may have isometric duals (cf. [6], p. 179). It is also known that Lip(I, d^a) is isomorphic to the sequence space l_∞ for each $a \in (0, 1]$, where d is the Euclidean metric. This fact is proved in [1], Theorem 1, for $a < 1$ and follows from [7] for $a = 1$ (see the proof of Lemma 1). From [1] and [2], Theorems 4.5 and 4.7, it follows that $\Delta(I, d^a)$ is isomorphic to the sequence space l_1 for $a < 1$. In the present note we show that if S contains a copy of I under a Lipschitz homeomorphism, then $\Delta(S)$ contains a complemented subspace isomorphic to L^1, the absolutely summable functions on I. (A subspace F of E is said to be *complemented* if there is a continuous projection of E onto F.)

In [3] and [4], Theorem 1, it is shown that Lip(S) contains a subspace isomorphic to l_∞ if S is infinite. The construction there can be used to

* This paper was supported in part by an O.S.U. Arts and Sciences research grant.

show directly that $\Delta(S)$ contains a subspace isomorphic to l_1. However, a theorem of Bessaga and Pełczyński (see [6], I.2.17) allows us to conclude that $\Delta(S)$ contains a complemented subspace isomorphic to l_1 if S is infinite. It should also be remarked that $\Delta(S)$ need not be isomorphic to a space $L^1(X, A, \mu)$ (in fact, not even an $\mathscr{L}_1$-space in the sense of Lindenstrauss and Pełczyński). This was shown in [5] by taking S to be the Hilbert cube. Thus, the isomorphic classification of the spaces $\Delta(S)$ and $\mathrm{Lip}(S)$ is far from understood.

LEMMA 1. *$\Delta(I)$ is isomorphic to $L^1 \oplus \boldsymbol{R}$.*

Proof. Given $f \in \mathrm{Lip}(I)$, the derivative of f is in L^∞ and $\|f'\|_\infty = \|f\|_d$. Thus it follows that $Tf = (f', f(0))$ is an isomorphism of $\mathrm{Lip}(I)$ onto $L^\infty \oplus \boldsymbol{R}$. If T^* is the adjoint map, $[a, b] \subset I$ and $r \in \boldsymbol{R}$, then

$$T^*(\chi_{[a,b]}, r) = \varepsilon_b - \varepsilon_a + r\varepsilon_0,$$

where $\chi_{[a,b]}$ is the characteristic function of $[a, b]$. Hence, if g is a step function, then $T^*(g, r) \in \Delta(I)$. Furthermore, $T^*(\chi_{[0,b]}, 1) = \varepsilon_b$ for each $b \in [0, 1]$, so $T^*(L^1 \oplus \boldsymbol{R}) = \Delta(I)$. This completes the proof.

We will call a mapping H from I into S a *Lipschitz homeomorphism* if both H and H^{-1} satisfy a Lipschitz condition.

LEMMA 2. *If H is a Lipschitz homeomorphism of I into S, then $\Delta(I)$ is isomorphic to a complemented subspace of $\Delta(S)$.*

Proof. Given

$$\varphi = \sum_{j=1}^{n} \lambda_j \varepsilon_{x_j} \in \Delta(I),$$

put

$$T\varphi = \sum_{j=1}^{n} \lambda_j \varepsilon_{H(x_j)} \in \Delta(S).$$

T is easily seen to be bounded. By [8], Proposition 1.4, we can find a map $H': S \to I$ which extends H^{-1} without increasing the Lipschitz constant. Thus, for $g \in \mathrm{Lip}(I)$,

$$|\varphi(g)| = |\varphi(g \circ H' \circ H)| = |T\varphi(g \circ H')| \leqslant \|T\varphi\| \, \|g\| \, (\mathrm{const}).$$

Hence T extends to an isomorphism from $\Delta(I)$ into $\Delta(S)$. Similarly, the map

$$Q: \sum \lambda_j \varepsilon_{s_j} \to \sum \lambda_j \varepsilon_{H'(s_j)}$$

extends to all of $\Delta(S)$, so that QT is the identity on $\Delta(I)$. Thus, TQ is a projection of $\Delta(S)$ onto $T(\Delta(I))$.

These two lemmas yield the following

PROPOSITION. *If there is a Lipschitz homeomorphism of I into S, then $\Delta(S)$ contains a complemented subspace isomorphic to L^1.*

We obtain two immediate corollaries.

COROLLARY 1. *If there is a Lipschitz homeomorphism of I into S, hen $\Delta(S)$ does not have the Radon-Nikodym property.*

COROLLARY 2. *If S is separable and there is a Lipschitz homeomorphism of I into S, then $\Delta(S)$ is not isomorphic to a dual space.*

As far as I know, the following question is open:

Is $\mathrm{Lip}(\boldsymbol{R}, d)$ isomorphic to l_∞, where $(\boldsymbol{R}, d)$ is the real line with the Euclidean metric? (**P 1056**)

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Reçu par la Rédaction le 27. 9. 1975;
en version modifiée le 13. 4. 1977