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INFINITARY VARIETIES OF STRUCTURES CLOSED UNDER THE FORMATION OF COMPLEX STRUCTURES

BY

G. GRÄTZER (WINNIPEG, MANITOBA)
AND S. WHITNEY* (EDMONTON, ALBERTA)

1. Results. Let $\mathfrak{A} = \langle A;\, F, R\rangle$ be a structure. Let Com A denote the power set of A and $\mathrm{Com}^+ A = \mathrm{Com}\, A - \{\emptyset\}$. For $f \in F$, define

$$f(A_0, \ldots) = \{f(a_0, \ldots) \mid a_0 \in A_0, \ldots\},$$

where $A_0, \ldots \in \mathrm{Com} - A$. For $r \in R$, define:

$r(A_0, \ldots)$ holds iff for any i and $b \in A_i$ there are

$$a_0 \in A_0, \ \ldots, \ a_i \in A_i, \ \ldots$$

such that $a_i = b$ and $r(a_0, \ldots, a_i, \ldots)$ holds.

(Observe that this agrees with the definition of set equality on Com A.) Let Com $\mathfrak{A}$ denote the resulting structure $\langle \mathrm{Com}\, A;\, F, R\rangle$. Then $\langle \mathrm{Com}^+ A;\, F, R\rangle$ is a substructure; it will be denoted by $\mathrm{Com}^+ \mathfrak{A}$. (These definitions agree with the usual definitions given in the finitary case in the literature; see the References. This definition of relations on Com $\mathfrak{A}$ appeared first in [6].)

A *variety* of structures is a class defined by a set of *atomic formulae*, i.e., formulae of the form $r(\boldsymbol{p}_0, \ldots)$, where $r \in R$ or r is the equality sign $=$, and $\boldsymbol{p}_0, \ldots$ are polynomials; if r is the equality sign, $r(\boldsymbol{p}_0, \boldsymbol{p}_1)$ stands for $\boldsymbol{p}_0 = \boldsymbol{p}_1$.

A polynomial $\boldsymbol{p}$ is *linear* iff every variable appearing in $\boldsymbol{p}$ occurs exactly once. An atomic formula $r(\boldsymbol{p}_0, \ldots)$ is *linear* iff $\boldsymbol{p}_0, \ldots$ are all linear; it is *regular* iff the same set of variables occurs in all $\boldsymbol{p}_i$. Now we can state the characterization theorem for varieties closed under Com.

THEOREM 1. *A variety* $\boldsymbol{K}$ *is* Com *closed* (*that is, if* $\mathfrak{A} \in \boldsymbol{K}$, *then* Com $\mathfrak{A} \in \boldsymbol{K}$) *iff it is definable by regular linear atomic formulae.*

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To state the analogous result for Com^+ closed varieties we need one more concept. Let K be a variety and let $\Phi = r(p_0, \ldots)$ ($r \in R$ or r is the equality sign) be an atomic formula. Φ is *almost linear in K* iff for any ordinal n less than the arity of r there is an atomic formula $\Psi = r(u_0, \ldots, u_n, \ldots)$ holding in K such that u_i is linear for $i \leqslant n$ and Φ can be obtained from Ψ by identifying variables (that is, substituting variables by variables).

THEOREM 2. *A variety K is* Com^+ *closed iff it is definable by a set of atomic formulae each almost linear in K.*

Observe that if the relations are finitary, then "almost linear" can be replaced by "linear".

COROLLARY. *Let K be a variety with finitary relations. Then K is* Com^+ *closed iff it is definable by linear formulae.*

In general, the Corollary does not hold.

THEOREM 3. *There is a variety K that is closed under* Com^+ *but cannot be defined by linear formulae.*

Various special cases of these results appear in the literature. Theorem 1 for varieties of finitary algebras defined by a single identity appears in [2]; see also [4]. The Corollary for finitary algebras was proved in [1] and, independently, in [5].

2. The Basic Lemma. All the results are based on one lemma. Before formulating it we state an observation from [2].

LINEARITY LEMMA. *Let $\mathfrak{A} = \langle A; F, R\rangle$ be a structure and let $p(x_0, \ldots, x_n, \ldots)$ be a linear polynomial in which all variables $x_0, \ldots, x_n, \ldots$ do occur. Then*

$$p(A_0, \ldots, A_n, \ldots) = \{p(a_0, \ldots, a_n, \ldots) \mid a_0 \in A_0, \ldots, a_n \in A_n, \ldots\}.$$

Observe that this does not hold if p is not linear; for instance, if $\mathfrak{A} = \langle A; \cdot\rangle$ is a groupoid, $p(x, y) = x^2 \cdot y$, then

$$p(A_0, A_1) = \{a_0 \cdot a_0' \cdot a_1 \mid a_0, a_0' \in A_0, a_1 \in A_1\}.$$

Using the "linearization" $p^* = x_1 \cdot x_2 \cdot y$ of $p = x^2 \cdot y$ we see that

$$p(A_0, A_1) = p^*(A_0, A_0, A_1),$$

and to compute $p^*(A_0, A_0, A_1)$ we can use the Linearity Lemma.

Let us call, in general, p^* a *generalization* of p if p can be obtained from p^* by identifying variables. If, in addition, p^* is linear, we call it a *linearization* of p. This leads to the following

COROLLARY. *Let $t(x_0, \ldots, x_i, \ldots)$ be a linearization of $p(y_0, \ldots, y_n, \ldots)$. Then*

$$p(A_0, \ldots, A_i, \ldots) = \{t(b_0, \ldots, b_n, \ldots) \mid b_n \in A_i \textit{ for } y_n = x_i\}.$$

Now we come to the crucial lemma:

Basic Lemma. *Let $\boldsymbol{K}$ be a* Com$^+$ *closed variety. Let $\boldsymbol{r}(\boldsymbol{p}_0, \ldots, \boldsymbol{p}_n, \ldots)$ hold in $\boldsymbol{K}$. Then for any ordinal n less than the arity of $\boldsymbol{r}$ we can find polynomials $\boldsymbol{p}'_0, \ldots, \boldsymbol{p}'_i, \ldots$ such that*

(i) *$\boldsymbol{r}(\boldsymbol{p}'_0, \ldots, \boldsymbol{p}'_i, \ldots)$ holds in $\boldsymbol{K}$;*

(ii) *for all i, $\boldsymbol{p}'_i$ is a generalization of $\boldsymbol{p}_i$;*

(iii) *$\boldsymbol{p}'_n$ is linear.*

Proof. Let X be an infinite set of variables such that $|X|$ exceeds the arity of polynomials and relations. Let $\mathfrak{P}$ be the free algebra over X modulo the identities $\boldsymbol{p} = \boldsymbol{q}$ holding in $\boldsymbol{K}$. We define the relations $\boldsymbol{r}$ on $\mathfrak{P}$ by

$$\boldsymbol{r}(\boldsymbol{p}_0, \ldots) \text{ in } \mathfrak{P} \text{ iff } \boldsymbol{r}(\boldsymbol{p}_0, \ldots) \text{ holds in } \boldsymbol{K},$$

thus obtaining the structure $\mathfrak{P} = \langle P; F, R \rangle$. Obviously, $\mathfrak{P} \in \boldsymbol{K}$. Since $\boldsymbol{K}$ is Com$^+$ closed, Com$^+$ $\mathfrak{P} \in \boldsymbol{K}$. Thus $\boldsymbol{r}(\boldsymbol{p}_0(X_0, \ldots), \ldots)$ holds in Com$^+$ $\mathfrak{P}$ for all nonempty subsets $X_0, X_1, \ldots$ of X. Let

$$\boldsymbol{p}'_n(\boldsymbol{x}_0, \ldots, \boldsymbol{x}_i, \ldots, \boldsymbol{x}_{00}, \boldsymbol{x}_{01}, \ldots, \boldsymbol{x}_{i0}, \boldsymbol{x}_{i1}, \ldots)$$

be a linearization of $\boldsymbol{p}_n(\boldsymbol{x}_0, \ldots)$; we obtain $\boldsymbol{p}_n$ from $\boldsymbol{p}'_n$ by setting

$$\boldsymbol{x}_0 = \boldsymbol{x}_{00} = \boldsymbol{x}_{01} = \ldots, \ldots, \boldsymbol{x}_i = \boldsymbol{x}_{i0} = \boldsymbol{x}_{i1} = \ldots, \ldots$$

Let Z be the set of all variables occurring in $\boldsymbol{r}(\boldsymbol{p}_0, \ldots, \boldsymbol{p}'_n, \ldots)$. For each $\boldsymbol{x}_i \in Z$ not occurring in $\boldsymbol{p}'_n$ we choose a singleton $(\boldsymbol{x}_i)_0 = \{z_i\}$ in X; for each $\boldsymbol{x}_i$ in $\boldsymbol{p}_n$ we choose a set

$$(\boldsymbol{x}_i)_0 = \{z_{ij} \mid j = 0, \ldots\} \cup \{z_i\}$$

such that the correspondence $\boldsymbol{x}_{ij} \to z_{ij}$ is one-to-one in j; we can further assume that all the sets chosen are pairwise disjoint.

Since all these sets are nonempty,

$$r\big(p_0((\boldsymbol{x}_0)_0, (\boldsymbol{x}_1)_0, \ldots), \ldots, p_n((\boldsymbol{x}_0)_0, \ldots), \ldots\big)$$

holds in Com$^+$ $\mathfrak{P}$.

Obviously,

$$b = p'_n(z_0, z_1, \ldots, z_{00}, z_{01}, \ldots, z_{i0}, z_{i1}, \ldots)$$

belongs to $p_n((\boldsymbol{x}_0)_0, (\boldsymbol{x}_1)_0, \ldots)$; so, by the definition of the relation $\boldsymbol{r}$ on Com$^+$ $\mathfrak{P}$, there are elements $a_i \in p_i((\boldsymbol{x}_0)_0, (\boldsymbol{x}_1)_0, \ldots)$ such that $r(a_0, \ldots, a_i, \ldots)$ in $\mathfrak{P}$ and $a_n = b$.

By the Corollary to the Linearity Lemma, there are linear polynomials

$$\boldsymbol{p}''_0, \ldots, \boldsymbol{p}''_{n-1}, \boldsymbol{p}''_{n+1}, \ldots$$

such that, for $i \neq n$, $a_i = \boldsymbol{p}''_i(\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots)$, where $\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots \in X$ and $\boldsymbol{p}''_i$ is a linearization of $\boldsymbol{p}_i$. For $i \neq n$, define $\boldsymbol{p}'_i = \boldsymbol{p}''_i(\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots)$. Since the $\boldsymbol{x}_i$ are not necessarily distinct, p_i' is not necessarily linear. However, no two distinct variables of p_i are equated, hence p_i' is a generalization of p_i. For p_n, we choose p_n' so that there is no identification, hence p_n' is linear. This completes the proof of the lemma.

3. Proof of results. To prove Theorem 1, let K be a variety defined by regular linear atomic formulae.

If $\mathfrak{A} \in K$ and $A_0, \ldots, A_n, \ldots$ are nonempty subsets of A, then (by the Linearity Lemma) any element of $p_n(A_0, \ldots)$ is of the form $p_n(a_0, \ldots, a_n, \ldots)$, where $a_i \in A_i$ for all i. Thus $r(p_0(A_0, \ldots), \ldots)$ is verified with the elements $p_i(a_0, \ldots) \in p_i(A_0, \ldots)$. This proves that K is closed under Com$^+$. Now, if the formulae are regular and if at least one of the A_j's is empty, then by regularity all $p_i(A_0, \ldots)$ are empty and $r(\emptyset, \ldots, \emptyset, \ldots)$ holds by definition. This shows that K is closed under Com.

Conversely, let K be closed under Com. Then every formula $r(p_0, \ldots)$ holding in K is regular. Indeed, if, say, x_i occurs in p_j but not in p_k $(j \neq k)$, then take $\mathfrak{A} \in K$ and set $x_i = \mathrm{O}$ and $x_n = A$ for all $n \neq i$. Then $p_i = \mathrm{O}$ and $p_k \neq \mathrm{O}$, contradicting $r(p_0, \ldots)$ for Com $\mathfrak{A}$.

Now let $r(p_0, \ldots)$ hold in K. Let us apply the Basic Lemma to K, $r(p_0, \ldots)$, and $n = 0$. By the Basic Lemma, we can assume that p_0 is linear. As we noted above, $r(p_0, \ldots)$ is regular. Hence p_0 and p_i have the same variables. This implies that p_i is also linear; indeed, if it is not, and, say, x_j occurs in p_i more than once, then the Basic Lemma gives us an $r(p_0, \ldots p_n', \ldots)$ holding in K in which x_j is replaced by at least two x_{jk}, contradicting that $r(p_0, \ldots, p_n', \ldots)$ must also be regular.

Thus for every $\Phi = r(p_0, \ldots)$ holding in K we found a linear regular $\Phi' = r(p_0', \ldots)$ from which Φ can be derived. Hence K can be defined by linear regular atomic formulae. This completes the proof of Theorem 1.

To prove Theorem 2, let K be defined by almost linear atomic formulae. Let $\mathfrak{A} \in K$: we claim that Com$^+$ $\mathfrak{A} \in K$. Indeed, if $r(p_0, \ldots)$ is an almost linear formula holding in K, we have to show that $r(p_0(A_0, \ldots), \ldots)$ holds in Com$^+$ $\mathfrak{A}$ for nonempty $A_0, \ldots \subseteq A$. Let $n \geq 0$ be fixed. By almost linearity, there is an $r(p_0', \ldots, p_n', \ldots)$ holding in K such that p_i' is a generalization of p_i and p_n' is linear. Thus any $b \in p_n'(A_0, \ldots)$ is of the form $b = p_n'(a_0, \ldots)$, where $a_i \in A_i$ for all i. Thus for $c_i = p_i'(a_0, \ldots)$ in $p_i'(A_0, \ldots)$ we have $r(c_0, \ldots, b, \ldots)$ in $\mathfrak{A}$, showing $r(p_0', \ldots, p_n', \ldots)$ in Com$^+$ $\mathfrak{A}$. However, $r(p_0, \ldots)$ is derivable from $r(p_0', \ldots)$, hence $r(p_0, \ldots)$.

The converse statement follows from our Basic Lemma by observing that a generalization of a linear polynomial is also linear. This completes the proof of Theorem 2.

4. An example. Let r be an ω-ary relation and let $\cdot$ be a binary operation: we write xy for $x \cdot y$ and x^2 for xx.

Let K be the variety defined by the following set of atomic formulae:

$$r(x_0^2, x_2^2, x_4^2, \ldots),$$
$$r(x_0 x_1, x_2^2, x_4^2, \ldots),$$
$$r(x_0 x_1, x_2 x_3, x_4^2, \ldots),$$
$$\ldots\ldots\ldots\ldots\ldots$$

These formulae establish that $r(x_0^2, x_2^2, x_4^2, \ldots)$ (and all the others) is almost linear. Hence, by Theorem 2, K is Com$^+$ closed.

To show that K cannot be defined by linear atomic formulae, take $X = \{x_0, x_1, \ldots\}$. Let G be the free groupoid over X. We define r on G as follows:

$r(g_0, g_1, \ldots)$ (for $g_0, g_1, \ldots \in G$) iff there is an $n \geqslant 0$ such that $g_m = (g'_m)^2$ for all $m \geqslant n$.

Obviously, $\mathfrak{G} = \langle G; \cdot, r\rangle \in K$.

We claim that there is no linear atomic formula holding in K, except those of the form $p = p$, where p is a linear polynomial. Indeed, if p and q are distinct groupoid polynomials, then $p \neq q$ in $\mathfrak{G}$. Hence $p = q$ does not hold. If $r(q_0, q_1, \ldots)$ is linear and holds in K, then $r(q_0, q_1, \ldots)$ must hold in $\mathfrak{G}$. Thus, by the definition of r in $\mathfrak{G}$, for some n we have $q_n = (q'_n)^2$, contradicting linearity. This completes the proof of Theorem 3.

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UNIVERSITY OF MANITOBA
WINNIPEG, MANITOBA

UNIVERSITY OF ALBERTA
EDMONTON, ALBERTA

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