COLLOQUIUM MATHEMATICUM

VOL. XXXII 1975 FASC. 2

AN EXTENSION OF APPROXIMATION THEOREMS

BY

JIŘÍ MOČKOŘ (BRNO)

In [7] Krull proved an approximation theorem for valuations of a field. This theorem was subsequently generalized by Jaffard [6] and Fukawa [1]. Some generalizations of the approximation theorem and of the independence theorem were given by Ribenboim [8]. Griffin [5] proved that if a family of valuations defining an integral domain is of finite character, the weak approximation theorem and approximation theorem are equivalent for a defining family for a ring of Krull type. Thus, in this case, the weak approximation theorem and approximation theorem also respect the integral closure of a domain in a finite algebraic extension of the quotient field. The motivation for this paper is to determine to what extent this property holds for an arbitrary family of valuations. Our results concern especially the case of the algebraic closure of the quotient field.

In what follows A denotes a commutative integral domain with identity and K denotes the quotient field of A. A family $\mathscr{F}$ of valuations of the field K is said to be of *finite character* if all but a finite number of valuations of $\mathscr{F}$ are zero at each non-zero element of K. If R_w is the valuation ring of K corresponding to a valuation $w \in \mathscr{F}$ and if $M(w)$ is a maximal ideal of R_w, then $\mathscr{F}$ is said to be a *defining family for A* provided that

$$A = \bigcap_{w \in \mathscr{F}} R_w$$

and $P(w) = M(w) \cap A$ is a prime ideal. The ideal $P(w)$ is then the *centre* of w on A. If $A_{P(w)} = R_w$, where $A_{P(w)}$ denotes the ring of fractions, then w is called an *essential valuation*. Let w, w' be valuations of K with the corresponding groups $\Gamma_w, \Gamma_{w'}$ and rings $R_w, R_{w'}$. If $R_w \subseteq R_{w'}$, we say that w' is *coarser* than w and write $w' \leqslant w$. If $w' \leqslant w$, then, for a prime ideal P of R_w, $R_{w'} = (R_w)_P$ and $\Gamma_{w'} \cong \Gamma_w/\Delta$, where Δ is the isolated subgroup of Γ_w generated by the values of those elements of R_w which do not belong to P. Since the set of all valuations coarser than w is totally ordered, we can find a finite valuation $w'' = w \wedge w' = \inf\{w, w'\}$. Let

Δ, Δ' be the corresponding isolated subgroups of Γ_w, $\Gamma_{w'}$, and let

$$\varphi : \Gamma_w \to \Gamma_w/\Delta \quad \text{and} \quad \varphi' : \Gamma_{w'} \to \Gamma_{w'}/\Delta'$$

be the canonical homomorphisms. Then we have $\Gamma_{w''} \cong \Gamma_{w'}/\Delta' \cong \Gamma_w/\Delta$ and we shall identify these three groups. A pair $(a, a') \in \Gamma_w \times \Gamma_{w'}$ is called *compatible* if $\varphi(a) = \varphi'(a')$.

Let $(w_i)_{i \in I}$ be a family of valuations of K with groups Γ_{w_i}, $i \in I$. Then $(a_i)_{i \in I} \in \prod_{i \in I} \Gamma_{w_i}$ is called *compatible* provided that every pair (a_i, a_j) in $\Gamma_{w_i} \times \Gamma_{w_j}$ is compatible. Instead of $\varphi(a)$ we shall usually write $_{(w,w')}\bar{a}$ (see [5]). Let $\mathscr{F}$ be a defining family of valuations for A. Let $w \in \mathscr{F}$ and let $a \in \Gamma_w$ with $a \neq 0$. Denote by Δ the largest isolated subgroup of Γ_w such that $a \notin \Delta$, and let $v \leqslant w$ be the valuation with group Γ_w/Δ. Then the set $\{w' \in \mathscr{F} : v \leqslant w'\}$ is denoted by $\mathscr{F}(a)$. We say that w is *well centred* on A if, for each $a \in \Gamma_w^+$, there exists an $a \in A$ such that $w(a) = a$. We say that w is *weakly independent* of $\mathscr{F}$ if, for $w' \in \mathscr{F}$ and $a' \in \Gamma_{w'}$ with $w \notin \mathscr{F}(a')$, there exists an $a \in A$ such that $w(a) = 0$ and $w'(a) \geqslant a'$. A family $\mathscr{F}$ is said to be *well centred* if every $w \in \mathscr{F}$ is well centred, and it is said to be *weakly independent* if every $w \in \mathscr{F}$ is weakly independent of $\mathscr{F}$. A defining family $\mathscr{F}$ for A is said to satisfy the *weak approximation theorem* (shortly, W.A.T.) if, for any finite number of valuations $w_i \in \mathscr{F}$, $i = 1, \ldots, n$, and for any compatible family

$$(a_1, \ldots, a_n) \in \prod_{i=1}^{n} \Gamma_{w_i}^+,$$

there exists an $a \in A$ such that $w_i(a) = a_i$ for $i = 1, \ldots, n$. Continuing with the same notation, the family

$$(a_w) \in \prod_{w \in \mathscr{F}'} \Gamma_w, \quad \text{where } \mathscr{F}' \subseteq \mathscr{F},$$

is called *complete with respect to* $\mathscr{F}$ if

$$\mathscr{F}' = \bigcup_{w \in \mathscr{F}'} \mathscr{F}(a_w).$$

A family $\mathscr{F}$ is said to satisfy the *approximation theorem* (shortly, A.T.) if, for any finite number of valuations $w_i \in \mathscr{F}$, $i = 1, \ldots, n$, and for a complete compatible family of elements

$$(a_1, \ldots, a_n) \in \prod_{i=1}^{n} \Gamma_{w_i},$$

there exists an $x \in K$ such that $w_i(x) = a_i$ for $i = 1, \ldots, n$, and $w(x) \geqslant 0$ for $w \in \mathscr{F}$, $w \neq w_i$, $i = 1, \ldots, n$.

Let A be integrally closed in K. If M is a maximal ideal of A, then $\mathscr{F}_M = \{w : w$ is a valuation of K positive on A and $P(w) = M\}$ is a defining family of A_M, where A_M denotes the ring of fractions with respect to the multiplicative system $A - M$. Let

$$\mathscr{F}^+ = \bigcup_M \mathscr{F}_M,$$

where M runs over the set of maximal ideals of A.

PROPOSITION 1. *Let A be integrally closed in K. If, for every maximal ideal M of A, $\mathscr{F}_M$ satisfies the weak approximation theorem, then $\mathscr{F}^+$ satisfies the weak approximation theorem.*

Proof. Let $\mathscr{F}_M$ satisfy the W.A.T. for every maximal ideal M of A. Then $\mathscr{F}_M$ is well centred on A_M and weakly independent (see [5], Proposition 5). It is clear that in this case $\mathscr{F}^+$ is well centred on A. Let $w, w' \in \mathscr{F}^+$ and let $a \in \Gamma_w$ be such that $w' \notin \mathscr{F}^+(a)$. There exists a maximal ideal M of A such that $P(w) = M$. Thus $w \in \mathscr{F}_M$. It is clear that $\mathscr{F}_M(a) \subseteq \mathscr{F}^+(a)$. Since $w \notin \mathscr{F}^+(a)$, it follows that $w \notin \mathscr{F}_M(a)$. Put

$$J = \{x \in A : w(x) \geqslant a\}.$$

Suppose the proposition is false. Then $x \in J$ implies that $w'(x) > 0$. It follows that $P(w) \subseteq P(w')$ (see the proof of Lemma 3 in [5]). But $P(w)$ is a maximal ideal of A, so that $P(w') = P(w) = M$. Thus $w' \in \mathscr{F}_M$. The family $\mathscr{F}_M$ being weakly independent, there exists an $a = a_1/a_2 \in A_M$ such that $w(a) \geqslant a$ and $w'(a) = 0$. Since $a_2 \in A - M = A - P(w) = A - P(w')$, it follows that $w'(a_1) = 0$ and $w(a_1) \geqslant a$, a contradiction with the assumption that $w'(a_1) > 0$ $(a_1 \in J)$. Therefore $\mathscr{F}^+$ is weakly independent and well centred. By Proposition 5 of [5], we infer that $\mathscr{F}^+$ satisfies the W.A.T.

PROPOSITION 2. *Suppose $\mathscr{F}$ is a defining family for A which satisfies the weak approximation theorem. Let $\mathscr{F}'$ be the family of canonical extensions of elements of $\mathscr{F}$ to valuations of $K(X)$, and let $\mathscr{G}$ denote the family of all valuations of $K(X)$ defined by irreducible polynomials from $K[X]$. Then $\mathscr{R} = \mathscr{F}' \cup \mathscr{G}$ satisfies the weak approximation theorem for the ring $A[X]$.*

Proof. It is well known that $\mathscr{R}$ is a defining family for $A[X]$. We shall show that $\mathscr{R}$ is weakly independent. Let $w', w_1' \in \mathscr{R}$ and let $a \in \Gamma_{w_1'}$ be such that $w' \notin \mathscr{R}(a)$. If $a \leqslant 0$, we have $w_1'(1_K) = 0$ and $w'(1_K) = 0 \geqslant a$. Thus we can assume that $a > 0$. There are four cases to be considered.

Case I. $w_1', w' \in \mathscr{G}$. If P and Q are the corresponding irreducible polynomials from $K[X]$, there exist irreducible polynomials $P', Q' \in A[X]$ and $p, q \in K$ such that $P = pP'$ and $Q = qQ'$. Since $a \in Z^+$, we have $a = (P')^a \in A[X]$, $w_1'(a) = a$ and $w'(a) = 0$.

Case II. $w_1', w' \in \mathscr{F}'$. Then we have $\Gamma_{w_1'} = \Gamma_{w_1}$ and $\Gamma_{w'} = \Gamma_w$, where $w_1 = w_1'/K$ and $w = w'/K$. Let Δ be the largest isolated subgroup of Γ_{w_1} such that $a \notin \Delta$ and let $v' \leqslant w_1'$ be the valuation of $K(X)$ with group Γ_{w_1}/Δ. Since X is a unit in $R_{w_1'}$, it follows that X is a unit in $R_{v'}$. Thus v' is the canonical extension of the valuation $v = v'/K$. Therefore

$$\Gamma_{v'} = \Gamma_v \quad \text{and} \quad \mathscr{F}(a) = \{w'' \in \mathscr{F}\colon w'' \geqslant v\}.$$

If $w \in \mathscr{F}(a)$, we have $w \geqslant v$. Now, since v' and w' are the canonical extensions of v and w, respectively, we get $w' \geqslant v'$. Thus $w' \in \mathscr{R}(a)$, a contradiction. Hence $w \notin \mathscr{F}(a)$. Since $\mathscr{F}$ satisfies the W.A.T., by Proposition 5 of [5], there exists an $a \in A \subset A[X]$ such that $w_1'(a) = w_1(a) \geqslant a$ and $w'(a) = w(a) = 0$.

Case III. $w_1' \in \mathscr{F}'$ and $w' \in \mathscr{G}$. We have $\Gamma_{w'} = Z$ and $\Gamma_{w_1'} = \Gamma_{w_1}$. By Lemma 1 of [5], we get

$$\mathscr{R}(a) = \{w'' \in \mathscr{R} : {}_{(w_1', w'')}\bar{a} \neq 0\}.$$

Since $w' \notin \mathscr{R}(a)$, it follows that ${}_{(w_1', w')}\bar{a} = 0$. The group Z being of rank 1, we see that $\Gamma_{w_1' \wedge w'} = Z$. Therefore $a = 0$, a contradiction. Thus this case does not occur.

Case IV. $w' \in \mathscr{F}'$ and $w_1' \in \mathscr{G}$. We have $\Gamma_{w'} = \Gamma_w$, $\Gamma_{w_1'} = Z$ and $0 \neq a \in Z$. By the assumption that $w' \notin \mathscr{R}(a)$, we get ${}_{(w_1', w')}\bar{a} = 0$. Therefore $a = 0$, a contradiction. Thus this case does not occur.

We have proved that the family $\mathscr{R}$ is weakly independent. The valuations of $\mathscr{G}$ are essential for $A[X]$, and hence they are well centred on $A[X]$. Since $\Gamma_{w'} = \Gamma_w$ for every $w' \in \mathscr{F}'$, where $w = w'/K$, it follows that $\mathscr{F}'$ is well centred. By Proposition 5 of [5], $\mathscr{R}$ satisfies the W.A.T.

PROPOSITION 3. *Suppose $\mathscr{F}$ is a defining family for A which satisfies the weak approximation theorem. Let L be the algebraic closure of the field K and let A' be the integral closure of A in L. Then the family $\mathscr{F}'$ of all extensions of valuations of $\mathscr{F}$ to valuations of L satisfies the weak approximation theorem for the ring A'.*

Proof. It is well known that $\mathscr{F}'$ is a defining family for A'. Let $w_1', \ldots, w_k' \in \mathscr{F}'$ and let

$$(a_1, \ldots, a_k) \in \prod_{i=1}^{k} \Gamma^{+}_{w_i'}$$

be a compatible family. Let $w_i = w_i'/K$, $i = 1, \ldots, k$. Since L/K is the algebraic extension, the factor group $\Gamma_{w_i'}/\Gamma_{w_i}$ is a torsion for $i = 1, \ldots, k$. There exists an $n \in Z^+$ such that

$$(na_1, \ldots, na_k) \in \prod_{i=1}^{k} \Gamma^{+}_{w_i}.$$

We shall show that this family is compatible. Let $1 \leqslant i,\ j \leqslant k$ and $i \neq j$. We have $w_i' \wedge w_j'/K = w_i \wedge w_j$. Indeed, it is clear that $w_i' \wedge w_j'/K \leqslant w_i, w_j$. If R is a valuation ring in K such that $R \supseteq R_{w_i}, R_{w_j}$, there exists a valuation ring R' in L lying over R and such that $R' \supseteq R_{w_i'}, R_{w_j'}$. Then $R' \supseteq R_{w_i' \wedge w_j'}$. It follows that $R \supseteq R_{w_i' \wedge w_j'} \cap K$, which yields

$$w_i' \wedge w_j'/K = w_i \wedge w_j.$$

If $\Gamma_{w_i' \wedge w_j'} = \Gamma_{w_i'}/\Delta_i'$, then

$$\Gamma_{w_i \wedge w_j} = \Gamma_{w_i}/(\Delta_i' \cap \Gamma_{w_i}).$$

Indeed, we have $R_{w_i' \wedge w_j'} = (R_{w_i'})_{B'}$ for a suitable prime ideal B' of $R_{w_i'}$. Hence the isolated subgroup Δ_i' of $\Gamma_{w_i'}$ is generated by the set $\{w_i'(x) : x \in R_{w_i'} - B'\}$. There exists a maximal ideal M of the integral closure $(R_{w_i})'$ of the ring R_{w_i} in the field L such that $R_{w_i'} = (R_{w_i})'_M$. Hence

$$(R_{w_i'})_{B'} = \big((R_{w_i})'_M\big)_{[B' \cap (R_{w_i})'_M](R_{w_i})'_M} = (R_{w_i})'_{[B' \cap (R_{w_i})']}.$$

Thus $R_{w_i \wedge w_j} = (R_{w_i})_{B' \cap R_{w_i}}$. If $\Gamma_{w_i \wedge w_j} = \Gamma_{w_i}/\Delta_i$ for a suitable isolated subgroup Δ_i of Γ_{w_i}, then Δ_i is generated by the set

$$\{w_i(x) : x \in R_{w_i} - B' \cap R_{w_i}\} = \{w_i'(x) : x \in R_{w_i} - B' \cap R_{w_i}\}.$$

This family also generates the isolated subgroup $\Delta_i' \cap \Gamma_{w_i}$. Hence $\Delta_i = \Delta_i' \cap \Gamma_{w_i}$.

Define a map f from $\Gamma_{w_i}/(\Delta_i' \cap \Gamma_{w_i})$ into $\Gamma_{w_i'}/\Delta_i'$ by setting

$$f(a + \Delta_i' \cap \Gamma_{w_i}) = a + \Delta_i'.$$

It is clear that f is a monomorphism. Let us consider the diagram

$$\begin{tikzcd}
\Gamma_{w_i'} \arrow[r, two heads, "\varphi_i'"] & \Gamma_{w_i'}/\Delta_i' \arrow[r, two heads, "="] & \Gamma_{w_i' \wedge w_j'} \\
\Gamma_{w_i} \arrow[u, "\mathrm{id}_{\Gamma_{w_i}}"] \arrow[r, two heads, "\varphi_i"] & \Gamma_{w_i}/\Delta_i' \cap \Gamma_{w_i} \arrow[u, "f"] \arrow[r, "="] & \Gamma_{w_i \wedge w_j} \arrow[u]
\end{tikzcd}$$

where φ_i and φ_i' are the canonical homomorphisms. Since

$$\Gamma_{w_i}/\Delta_i' \cap \Gamma_{w_i} = \Gamma_{w_j}/\Delta_j' \cap \Gamma_{w_j} \quad \text{and} \quad \Gamma_{w_i'}/\Delta_i' = \Gamma_{w_j'}/\Delta_j',$$

we have an analogous diagram for the index j. Now, (a_i, a_j) being compatible, it follows that $\varphi_i'(na_i) = \varphi_j'(na_j)$. Since the left square in the diagram is commutative,

$$f(\varphi_i(na_i)) = \varphi_i'(na_i) = \varphi_j'(na_j) = f(\varphi_j(na_j)).$$

Since f is a monomorphism, we have $\varphi_i(na_i) = \varphi_j(na_j)$. Therefore the pair (na_i, na_j) is compatible.

The family $\mathscr{F}$ satisfying the W.A.T., it follows that there exists an $a \in A$ such that $w_i(a) = na_i$ for $i = 1, \ldots, k$. Since L is the algebraic closure of K, there exists a $b \in A'$ such that $a = b^n$. Therefore $a_i = w_i'(b)$ for $i = 1, \ldots, k$, i. e. $\mathscr{F}'$ satisfies the W.A.T. for the ring A'.

PROPOSITION 4. *Let $\mathscr{F}$ be a defining family for A which satisfies the approximation theorem. Let L be the algebraic closure of the field K. Then the family $\mathscr{F}'$ of all extensions of valuations of $\mathscr{F}$ to valuations of L satisfies the approximation theorem.*

Proof. Let $w_1', \ldots, w_k' \in \mathscr{F}'$ and let

$$(a_1, \ldots, a_k) \in \prod_{i=1}^{k} \Gamma_{w_i'}$$

be a complete compatible family. Put $w_i = w_i'/K$ for $i = 1, \ldots, k$. Since L/K is the algebraic extension, the factor group $\Gamma_{w_i'}/\Gamma_{w_i}$ is a torsion for $i = 1, \ldots, k$. There exists an $n \in Z^+$ such that

$$(na_1, \ldots, na_k) \in \prod_{i=1}^{k} \Gamma_{w_i}.$$

As in the proof of Proposition 3, we infer that $(na_1, \ldots, na_k)$ is a compatible family. We shall show that it is complete.

The family $(a_1, \ldots, a_k)$ is complete, so that

$$\{w_1', \ldots, w_k'\} = \bigcup_{i=1}^{k} \mathscr{F}'(a_i).$$

Thus, for every i, $1 \leqslant i \leqslant k$, there exists a j_i, $1 \leqslant j_i \leqslant k$, such that $w_i' \in \mathscr{F}'(a_{j_i})$ (if $a_i \neq 0$, we have $w_i' \in \mathscr{F}'(a_i)$). It is clear that $\mathscr{F}'(a_{j_i}) = \mathscr{F}'(na_{j_i})$. As in the proof of Proposition 3, it follows that

$$\Gamma_{w_i \wedge w_{j_i}} = \Gamma_{w_{j_i}}/\Delta' \cap \Gamma_{w_{j_i}}, \quad \text{where } \Gamma_{w_{j_i}' \wedge w_i'} = \Gamma_{w_{j_i}'}/\Delta'.$$

Since $w_i' \in \mathscr{F}'(na_{j_i})$, we have $na_{j_i} \notin \Delta' \cap \Gamma_{w_{j_i}}$. It follows that ${}_{(w_{j_i}, w_i)}\overline{na}_{j_i} \neq 0$, so that $w_i \in \mathscr{F}(na_{j_i})$. Therefore

$$\{w_1, \ldots, w_k\} \subseteq \bigcup_{i=1}^{k} \mathscr{F}(na_i).$$

Let $w \in \mathscr{F}(na_i)$ and let $w' \in \mathscr{F}'$ be an extension of the valuation w. Then we have $w' \in \mathscr{F}'(na_i)$. In fact, if

$$\Gamma_{w_i \wedge w} = \Gamma_{w_i}/\Delta \quad \text{and} \quad \Gamma_{w'_i \wedge w'} = \Gamma_{w'_i}/\Delta',$$

then $\Delta = \Delta' \cap \Gamma_{w_i}$. Since $na_i \notin \Delta$, we have $w' \in \mathscr{F}'(na_i) = \mathscr{F}'(a_i)$ (see [5], Lemma 1). Since $(a_1, \ldots, a_k)$ is complete, there exists a j, $1 \leqslant j \leqslant k$, such that $w' = w'_j$, so that $w = w_j$.

The family $\mathscr{F}$ satisfying the A.T., it follows that there exists an $a \in K$ such that $w_i(a) = na_i$ for $i = 1, \ldots, k$, and $w(a) \geqslant 0$ for $w \in \mathscr{F}$, $w \neq w_i$ for $i = 1, \ldots, k$. Since L/K is the algebraic closure of the field K, there exists a $b \in L$ such that $b^n = a$. Thus $w'_i(b) = a_i$ for $i = 1, \ldots, k$. If $w' \in \mathscr{F}'$ and $w' \neq w'_i$, $i = 1, \ldots, k$, then $w = w'/K \in \mathscr{F}$ and $w \neq w_i$, $i = 1, \ldots, k$. It follows that $w'(b) \geqslant 0$. Therefore $\mathscr{F}'$ satisfies the A.T.

REFERENCES

[1] M. Fukawa, *On the theory of valuations*, Journal of the Faculty of Science, University of Tokyo, Section I, 12 (1965), p. 57-79.

[2] R. Gilmer, *Multiplicative ideal theory*, New York 1972.

[3] M. Griffin, *Families of finite character and essential valuations*, Transactions of the American Mathematical Society 130 (1968), p. 75-85.

[4] — *Some results on v-multiplicative rings*, Canadian Journal of Mathematics 19 (1967), p. 710-722.

[5] — *Rings of Krull type*, Journal für die reine und angewandte Mathematik 229 (1968), p. 1-27.

[6] P. Jaffard, *Théorie arithmétique des anneaux du type de Dedekind, I*, Bulletin de la Société Mathématique de France 80 (1952), p. 61-100.

[7] W. Krull, *Allgemeine Bewertungstheorie*, Journal für die reine und angewandte Mathematik 167 (1932), p. 160-196.

[8] P. Ribenboim, *Le théorème d'approximation pour les valuations de Krull*, Mathematische Zeitschrift 68 (1957), p. 1-18.

Reçu par la Rédaction le 3. 9. 1973;
en version modifiée le 13. 12. 1973