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TOPOLOGICAL GROUPS OF DIVISIBILITY

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Let A be an integral domain with the quotient field K. Then the group of divisibility $G(A)$ of A is the partially ordered group $K^*/U(A)$, where K^* denotes the multiplicative group of K and $U(A)$ the group of units of A with $aU(A) \leqslant bU(A)$ if and only if a divides b in A. It is well known (see [10]) that any Abelian lattice-ordered group is a group of divisibility of some Bézout domain. (For analogous problems see [3].)

But every lattice-ordered group may be endowed with the discrete topology and, therefore, considered as a topological lattice-ordered group. We have an analogous situation for fields: every field may be considered as a topological field with respect to the discrete topology.

Hence, it seems natural to consider the following question: do there exist, for any Abelian lattice-ordered topological group G, a topological field K and a Bézout domain A in K such that $U(A)$ is closed in K^* with respect to the topology induced from K and such that the factor group $K^*/U(A)$ is a topological lattice-ordered group isomorphic (i.e. group and lattice homeomorphic) to G?

The aim of this note is to solve this problem especially for a topological lattice-ordered group such that there exists a homeomorphism from this group into the Cartesian product of totally ordered topological groups.

All groups and rings are assumed to be commutative.

At first, we recall some basic facts from the theory of topological lattice-ordered groups. A *topological lattice-ordered group* (notation: tl-group) is a triple $(G, \leqslant, \mathcal{T})$ (henceforth denoted simply by G), where G is a group, $\leqslant$ is a partial order, and $\mathcal{T}$ is a topology on the underlying set $|G|$ of G such that $(G, \leqslant)$ is a lattice-ordered group (notation: l-group), $(G, \mathcal{T})$ is a topological group, and $(|G|, \leqslant, \mathcal{T})$ is a topological lattice. By $\mathcal{T}(0)$ we denote the complete system of neighbourhoods of zero in G. We say that two tl-groups are *tl-isomorphic* if there is a homeomorphism between them which is both a lattice and group isomorphism.

Let G be a tl-group, and $\mathcal{T}$ its topology. If the sets $(-\infty, g) = \{x \in G: x < g\}$ and $(g, \infty) = \{x \in G: x > g\}$ are open for any $g \in G$, then $\mathcal{T}$ is called a *semi-interval topology* (notation: si-topology, see [16]).

Let G be an l-group. We use G^+ to denote $\{g \in G: g \geqslant 0\}$, where 0 is the zero of G. A *prime l-ideal* P of G is a convex subgroup of G which is also a sublattice, and if $a \wedge b \in P$, then $a \in P$ or $b \in P$ (here $a \wedge b = \inf\{a, b\}$ in G) for any $a, b \in G$. If $\{G_i: i \in J\}$ is a set of l-groups, then the *direct product* $\prod_{i \in J} G_i$ is the set of all functions f on J such that $f(i) \in G_i$, the operations in $\prod_{i \in J} G_i$ being performed componentwise. An l-group G is said to be a *subdirect sum* of $\{G_i: i \in J\}$ if G is an l-subgroup of $\prod_{i \in J} G_i$ and each projection map $\pi_i: G \to G_i$ is a surjection.

Let G be a tl-group and let $\{P_i: i \in J\}$ be a collection of closed prime l-ideals of G which meet in the zero. We say that $\{P_i: i \in J\}$ is a *topological realization* of G if the natural map

$$\pi: G \to \prod_{i \in J} (G/P_i)$$

is a homeomorphism from G onto πG, where πG inherits its topology from $\prod_{i \in J} (G/P_i)$ (see [8]).

Further, let K be a topological field with topology $\mathcal{T}$. Then the multiplicative group K^* of K is a topological group with respect to the topology induced from K. Now, let w be a valuation on a field K with the value group G_w. Then a valuation w defines a field topology $\mathcal{T}_w$ in K: we take the sets of the form

$$U_{w,a} = \{x \in K: w(x) > a\}, a \in G_w^+, \quad \text{and} \quad R_w = \{x \in K: w(x) \geqslant 0\}$$

as a base of the neighbourhoods of zero in K. If G_w is considered to be a discrete topological group, then w is continuous. We say that a family of valuations $\{w_i: i \in J\}$ on a field K is a *defining family* for an integral domain $A \subseteq K$ if

$$A = \bigcap_{i \in J} R_{w_i}.$$

We recall the notion of a locally bounded topology. Let K be a topological field. A subset $B \subset K$ is said to be *bounded* if for every neighbourhood V of zero there exists another neighbourhood U with $BU \subseteq V$. Then K is said to be *locally bounded* provided there exists an open bounded set in K.

Finally, we say that a triple $(K, \mathcal{T}, A)$ is a *representation* of a tl-group G if K is a topological field with a topology $\mathcal{T}$, A is a Bézout domain with the quotient field K, the group of units $U(A)$ of A is closed in K^*, and the topological factor group $G(A) = K^*/U(A)$ is a tl-group which is tl-isomorphic to G. In this case we say that *G has a representation.*

It should be observed that in many cases we need not investigate whether $G(A)$ is a topological lattice. Indeed, the following simple lemma holds:

LEMMA 1. *Let $(G, \leqslant, \mathcal{T})$ be a tl-group and let G_1 be a topological group with a topology $\mathcal{T}_1$ such that G_1 is an l-group. If there exists a homeomorphism ψ of G_1 onto G such that it is an l-isomorphism, then G_1 is a tl-group.*

Proof. Let $U \in \mathcal{T}_1(0)$ and $g \in G_1$. Since G is a tl-group, there exists $V \in \mathcal{T}_1(0)$ such that

$$\big(\psi V - (\psi g)^+\big) \vee \big(\psi V + (\psi g)^-\big) \subseteq \psi U,$$

where $x^+ = x \vee 0$ and $x^- = x \wedge 0$. Now, since ψ is an l-isomorphism, we obtain

$$(V - g^+) \vee (V + g^-) \subseteq U$$

and, by [14], Theorem 1.3, G_1 is a tl-group.

LEMMA 2. *Let K be a topological field with a topology $\mathcal{T}$. Then K^* is an open set in K. Further, if K is non-discrete, then K^* is a dense set in K.*

Proof. Let $x \in K^*$. Since K is a topological field, we may find a neighbourhood W of x in K such that $0 \notin W$. Hence, $x \in W \subseteq K^*$ and K^* is open in K. Further, let K be non-discrete. Then for any neighbourhood V of zero in K we have $V \cap K^* \neq \emptyset$. Thus, K^* is dense in K.

LEMMA 3. *Let K be a topological field with a topology $\mathcal{T}$ and let $w\colon K^* \to G_w$ be a valuation on K such that w is continuous with respect to the discrete topology on G_w. Then $\mathcal{T}_w \leqslant \mathcal{T}$.*

Proof. Since w is continuous, it follows that, for any $a \in G_w$ and $x_a \in K^*$ such that $w(x_a) = a$, the set

$$w^{-1}(a) = x_a + U_{w,a} \subseteq K^*$$

is open in K^*. From Lemma 2 we infer that $U_{w,a}$ is open in K. Hence $\mathcal{T}_w \leqslant \mathcal{T}$.

PROPOSITION 1. *Let H be a closed l-ideal of a tl-group G. If G has a representation, then the factor tl-group G/H has a representation.*

Proof. Let $(K, \mathcal{T}, A)$ be a representation of G. By [9], Theorem 2.1, there exists a saturated multiplicative system S in A such that the group of divisibility $G(A_S)$ of a quotient domain A_S is l-isomorphic to G/H. Hence, there exists an l-isomorphism σ which completes the commutative diagram

$$\begin{tikzcd}
K \arrow[r, "f"] \arrow[dr, "g"'] & G(A) \arrow[r, "\varphi"] & G \arrow[d, "s"] \\
 & G(A_S) \arrow[r, "\sigma"] & G/H
\end{tikzcd}$$

where f, g, and ε are the canonical homomorphisms, and φ is a tl-isomorphism associated with the representation $(K, \mathcal{T}, A)$. Since $U(A_S) = (\varphi f)^{-1}(H)$, $U(A_S)$ is closed in K^* and we may suppose that $G(A)$ and $G(A_S)$ are the factor topological groups of K^*. Now, let $(U+H)/H$ be open in G/H; then $(\varphi f)^{-1}(U)/U(A_S)$ is open in $G(A_S)$ and

$$\sigma\big((\varphi f)^{-1}(U)/U(A_S)\big) \subseteq (U+H)/H.$$

Thus, σ is continuous. Furthermore, since $U\cdot U(A_S)/U(A_S)$ is an open set in $G(A_S)$, it is easy to see that

$$\varepsilon\varphi f(U) = \sigma\big(U\cdot U(A_S)/U(A_S)\big)$$

is open in G/H. Hence, by Lemma 1, $(K, \mathcal{T}, A_S)$ is a representation of G/H.

The following theorem solves completely the problem of existence of a representation for a totally ordered tl-group.

THEOREM 1. *Let $(G, \leqslant, \mathcal{T})$ be a totally ordered tl-group. Then G has a representation if and only if G is a discrete space.*

Proof. Suppose that G is a discrete tl-group and let K be the quotient field of the group algebra $\mathcal{A}_k(G)$ of G over the fixed field k. Then the classical result of Krull asserts that there exists a valuation w on K with the value group G. Since $U(R_w)$ is open in $(K, \mathcal{T}_w)$, G has a representation.

Conversely, suppose that G has a representation $(K, \mathcal{T}, A)$. Then $w = \varphi f$: $K^* \to G$ is a valuation on K, where f is the canonical map of K^* onto $G(A)$ and φ is a tl-isomorphism associated with the representation. Thus, $R_w = A$. Now, by [16], 1.2, $\mathcal{T}$ is an si-topology, and since w is continuous, the sets

$$U_{w,a} = w^{-1}\big((a, \infty)\big), \qquad a \in G,$$

are open in K^*. By Lemma 2, $\mathcal{T}_w \leqslant \mathcal{T}$, and since $U(R_w) = U(A)$ is open in $\mathcal{T}_w$, G is a discrete space.

It is well known that the factor group of an l-group with respect to a prime l-ideal is totally ordered. Then, using Proposition 1 and Theorem 1, we obtain the following result:

COROLLARY 1. *If a tl-group G has a representation, then every closed prime l-ideal of G is open in G.*

Observe that, by Corollary 1, an example of a tl-group (not totally ordered) which has no representation is easy to construct. The example of a topological product of two copies of a totally ordered group with the interval non-discrete topology works.

PROPOSITION 2. *Let $(G, \leqslant, \mathcal{F})$ be a tl-group and let $(K, \mathcal{T}, A)$ be its representation. Then there exists a topological realization of G if and only if there is a defining family $\{w_i\colon i \in J\}$ for A such that $\mathcal{T} \geqslant \sup\{\mathcal{T}_{w_i}\colon i \in J\}$*

and $\{U(R_{w_i})\colon i \in J\}$ *is a subbase for the sets* $U \cdot U(A)$, *where* U *is an open neighbourhood of* 1_K *in* K^*.

Proof. Suppose that $\{w_i\colon i \in J\}$ is a defining family for A which satisfies the conditions of the proposition. The value group G_{w_i} of w_i, $i \in J$, may be considered to be a tl-group with respect to the discrete topology. Then for every $i \in J$ there is a continuous and open l-epimorphism ε_i which completes the commutative diagram

$$\begin{tikzcd} K \arrow[r, "f"] \arrow[d, "w_i"'] & G(A) \arrow[d, "\varphi"] \\ G_{w_i} & G \arrow[l, dashed, "\varepsilon_i"'] \end{tikzcd}$$

where f is the canonical homomorphism and φ is a tl-isomorphism associated with the representation. Indeed, for every $a \in G$ we put

$$\varepsilon_i(a) = w_i(x), \qquad \text{where } x \in \varphi^{-1}(a).$$

Since $U(A) \subseteq U_i = U(R_{w_i})$, the definition is correct.

Now, if $a \geqslant 0$, then for every $x \in \varphi^{-1}(a)$ we have $x \in A \subseteq R_{w_i}$, whence $\varepsilon_i(a) = w_i(x) \geqslant 0$. Conversely, since A is a Bézout domain, for any $\alpha \in G_{w_i}^+$ there exists $a \in A^* = A - \{0\}$ such that $w_i(a) = \alpha$ (see [5]). Then $\varphi f(a) \geqslant 0$ and $\varepsilon_i(\varphi f(a)) = \alpha$. Further, since $\mathscr{T} \geqslant \sup\{\mathscr{T}_{w_i}\colon i \in J\}$ by Lemma 3, $w_i^{-1}(U)$ is open for any $U \subseteq G_{w_i}$. Hence, $\varphi f(w_i^{-1}(U)) \subseteq \varepsilon_i^{-1}(U)$ is open in G, and ε_i is open and continuous.

Then for any $i \in J$ there exists a closed and open prime l-ideal H_i of G such that the factor tl-group G/H_i is tl-isomorphic to G_{w_i}. We shall identify these two groups.

Now, applying the identity $w_i = \varepsilon_i \varphi f$, we obtain $H_i = \varphi f(U_i)$, and since

$$U(A) = \bigcap_{i \in J} U_i,$$

we have

$$\bigcap_{i \in J} H_i = \{0\}.$$

Finally, let $\mathscr{U} \subseteq G$ be an open neighbourhood of zero in G. Then $\varphi^{-1}(\mathscr{U}) = U \cdot U(A)/U(A)$ for some open neighbourhood U of 1_K in K^*. By the assumption, there exist $i_1, i_2, \ldots, i_n \in J$ such that

$$\bigcap_{t=1}^{n} U_{i_t} \subseteq U \cdot U(A).$$

Hence

$$\bigcap_{t=1}^{n} H_{i_t} \subseteq \varphi f\Big(\bigcap_{t=1}^{n} U_{i_t}\Big) \subseteq \mathscr{U}.$$

Therefore, $\{H_i: i \in J\}$ is a subbase of neighbourhoods of zero in G which meet at zero and, by [8], Theorem 12, $\{H_i: i \in J\}$ is a topological realization of G.

Conversely, suppose that $\{H_i: i \in J\}$ is a topological realization of G and let w_i, $i \in J$, be the composition of the maps

$$K^* \xrightarrow{f} G(A) \xrightarrow{\varphi} G \xrightarrow{\pi} \pi G \xrightarrow{\varepsilon_i} G/H_i,$$

where f, π, and ε_i are the canonical homomorphisms. Since f is a semi-valuation (see [10]) and φ, π, and ε_i are l-homomorphisms, w_i is a valuation for every $i \in J$ (see [10]). It is clear that w_i is continuous and from the fact that $\{G/H_i: i \in J\}$ is a realization of G we infer that $\{w_i: i \in J\}$ is a defining family for A. By Corollary 1, G/H_i is a discrete tl-group.

Now, it is easy to see that for every $i \in J$ there exists a tl-isomorphism τ_i which completes the commutative diagram

$$\begin{tikzcd} G/H_i \arrow[rr, dashed, "\tau_i"] & & K^*/U(R_{w_i}) \\ G \arrow[u, "\varepsilon_i \pi"] \arrow[rr, "\varphi^{-1}"] & & G(A) \arrow[u, "f_i"'] \end{tikzcd}$$

where f_i is the canonical map. Let

$$\tau = \prod_{i \in J} \tau_i$$

be the (categorical) product of τ_i and put $\tau' = \tau/\pi G$, the restriction of τ on πG. Then τ' is a tl-isomorphism of πG onto

$$(*) \qquad \Big\{\big(xU(R_{w_i})\big) \in \prod_{i \in J} K^*/U(R_{w_i}): \ x \in K^*\Big\}.$$

Finally, we set $\psi = \tau' \cdot \pi \cdot \varphi$. Then ψ is a homomorphism of $G(A)$ onto $(*)$. Now, since w_i is continuous for every $i \in J$, by Lemma 3 we obtain

$$\mathcal{T} \geqslant \sup\{\mathcal{T}_{w_i}: \ i \in J\}.$$

Let U be an open neighbourhood of 1_K in K^*. Then $U \cdot U(A)/U(A)$ is open in $G(A)$ and $\psi\big(U \cdot U(A)/U(A)\big)$ is open in $\psi\big(G(A)\big)$. Hence, there exist $i_1, i_2, \ldots, i_n \in J$ such that

$$\psi\Big(\bigcap_{k=1}^{n} U(R_{w_{i_k}})/U(A)\Big) = \prod_{k=1}^{n} \big(U(R_{w_{i_k}})/U(R_{w_{i_k}})\big) \times \prod_{j \neq i_k} K^*/U(R_{w_j}) \cap \psi\big(G(A)\big)$$
$$\subseteq \psi\big(U \cdot U(A)/U(A)\big).$$

Hence,

$$\prod_{k=1}^{n} U(R_{w_{i_k}}) \subseteq U\cdot U(A)$$

and the proof is complete.

THEOREM 2. *Let $(G, \leqslant, \mathcal{T})$ be a tl-group with a topological realization $\{H_i\colon i \in J\}$. Then there exists a representation $(K, \mathcal{T}, A)$ of G such that $(K, \mathcal{T})$ is a locally bounded field and $U(A)$ is a bounded set if and only if J is a finite set and H_i is open for every $i \in J$.*

Proof. Suppose that J is a finite set and H_i is open for every $i \in J$. Then G is a discrete tl-group. Let A be a Bézout domain with the quotient field K such that $G(A)$ is l-isomorphic to G and let

$$w_i = \varepsilon_i\cdot\pi\cdot\varphi\cdot f, \quad i \in J,$$

be the same as in the proof of Proposition 2. We set $\mathcal{T} = \sup\{\mathcal{T}_{w_i}\colon i \in J\}$. Then, by [7], $(K, \mathcal{T})$ is a locally bounded topological field. Since

$$A = \bigcap_{i\in J} R_{w_i} \quad \text{and} \quad U(A) = \bigcap_{i\in J} U(R_{w_i}),$$

$U(A)$ is open in K and $(K, \mathcal{T}, A)$ is a locally bounded representation of G. It remains to show that $U(A)$ is a bounded set. Let

$$\mathcal{U} = \bigcap_{i=1}^{n} U_{w_i,a_i}, \quad a_i \in G^+_{w_i}.$$

We may assume that the valuations w_i, $i \in J$, are mutually independent. By the approximation theorem for independent valuations, there exists an element $x \in K^*$ such that

$$w_i(x) > a_i, \quad i = 1, 2, \ldots, n.$$

Then $xU(A) \subseteq \mathcal{U}$, since $U(A) \subseteq U(R_{w_i})$, $i \in J$. Hence, by [1], chapitre 3, § 6, exemple 20, $U(A)$ is a bounded set in K.

Conversely, let $(K, \mathcal{T}, A)$ be a locally bounded representation of G such that $U(A)$ is a bounded set. Thus, we may find a bounded neighbourhood U of zero in K and, therefore, $\big((1+U)\cap K^*\big)\cdot U(A)$ is a bounded set. By Proposition 2, there exists a defining family $\{w_i\colon i \in J\}$ for A such that $\mathcal{T} \geqslant \sup\{\mathcal{T}_{w_i}\colon i \in J\}$, and $\{U(R_{w_i})\colon i \in J\}$ is a subbase for sets $U\cdot U(A)$, where U is a neighbourhood of 1_K in K^*. Hence, there are $i_1, i_2, \ldots, i_n \in J$ such that

$$\bigcap_{k=1}^{n} U(R_{w_{i_k}}) \subseteq \big((1+U)\cap K^*\big)\cdot U(A)$$

and it follows that $(K, \sup\{\mathcal{T}_{w_i}\colon i \in J\})$ is a locally bounded topological field. Again, by [7], J is a finite set and, by Corollary 1, H_i is open for every $i \in J$.

THEOREM 3. *Let* $(G, \leqslant, \mathscr{F})$ *be a tl-group with a topological realization* $\{H_i\colon i \in J\}$. *Then there exists a representation* $(K, \mathscr{T}, A)$ *of* G *such that* $(K, \mathscr{T})$ *is a locally compact field if and only if* G *is a discrete group isomorphic to the group* Z *of integers.*

Proof. Let $(K, \mathscr{T}, A)$ be a locally compact representation of G. By Proposition 2, there exists a defining family $\{w_i\colon i \in J\}$ for A such that $\sup\{\mathscr{T}_{w_i}\colon i \in J\} \leqslant \mathscr{T}$. Every locally compact topological field is a complete topological field and, by [18], Theorem 10, $\mathscr{T}$ is a minimal field topology in K. Since

$$\mathscr{T} \geqslant \sup\{\mathscr{T}_{w_i}\colon i \in J\} \geqslant \mathscr{T}_{w_i}, \quad i \in J,$$

$\mathscr{T}_{w_i} = \mathscr{T}_{w_j}$ for every $i, j \in J$. Thus, the valuations w_i, $i \in J$, are mutually dependent. Now, applying [2], § 5, Proposition 2, we infer that w_i is a discrete rank one valuation for every $i \in J$ and it follows that the valuations $w_i \neq w_j$ are mutually independent. Thus, $\operatorname{card} J = 1$ and $H_1 = \{0\}$. Therefore,

$$G \cong G/H_1 \cong G_{w_1} \cong Z.$$

Conversely, it remains to show that the discrete group Z has a locally compact representation. Consider a p-adic valuation v_p on the field Q of rational numbers for some prime number $p \in Z$. Then $G_{v_p} \cong Z$ and the completion Q_p of the topological field $(Q, \mathscr{T}_{v_p})$ is the field of p-adic numbers which is locally compact. Let $\hat{v}_p$ be the continuous extension of v_p on Q_p. Then $\hat{v}_p$ is a valuation on Q_p and it is clear that $(Q_p, \mathscr{T}_{\hat{v}_p}, R_{\hat{v}_p})$ is a locally compact representation of Z.

Further, we shall deal with a tl-group G such that the set of dual principal polars in G is a base of neighbourhoods of zero. Recall that two elements a and b of an l-group G are *disjunctive* if $|a| \wedge |b| = 0$. An element $g \in G$ is a *weak unit element* in G if the only disjunctive element in G to g is zero. Now, the *disjunctive complement* of a set $A \subseteq G$ is the set

$$A' = \{g \in G\colon |g| \wedge |a| = 0 \text{ for arbitrary } a \in A\}.$$

Further on, $A'' = (A')'$ and $A''' = A'$. Then the set $\{a\}'$ is called the *dual principal polar*. The set of all dual principal polars will be denoted by $\mathscr{P}$. We recall some basic facts about the completely regular realization of an l-group G. Let $\{G_i\colon i \in J\}$ be a realization of G. We introduce the following notation for $f \in G$ and $i \in J$ (see [12]):

$$Z(f) = \{i \in J\colon f(i) = 0\}, \quad \Psi(i) = \{f \in G\colon f(i) = 0\}.$$

Note that $\Psi(i)$ is a prime l-ideal and $G_i \cong G/\Psi(i)$. Now, a realization $\{G_i\colon i \in J\}$ is said to be *completely regular* if for any element $f \in G$ and any element $i \in Z(f)$ there exists an element $g \in G$ such that $i \in J - Z(g) \subseteq Z(f)$. According to [12], 8.7, a realization $\{G_i\colon i \in J\}$ is completely regular if and only if $\{\Psi(i): i \in J\}$ is a set of minimal prime l-ideals of G.

LEMMA 4. *Let $\{w_i: i \in J\}$ be a defining family for a Bézout domain A and let, for every $i \in J$, w_i be centred on a maximal ideal of A. Then $\{G_{w_i}: i \in J\}$ is a completely regular realization of $G(A)$.*

Proof. Since $\{w_i: i \in J\}$ is a defining family for A, $\{G_{w_i}: i \in J\}$ is a realization of the group $G(A)$. Now, by the assumption, for every $i \in J$ there exists a maximal ideal $\mathfrak{m}_i$ of A such that

$$R_{w_i} = A_{\mathfrak{m}_i}.$$

By [9], Corollary 2.4, there is a one-to-one correspondence between maximal ideals of A and ultrafilters in the positive cone $G(A)^+$ of $G(A)$. Indeed, if P and F correspond one to the other, then the group of divisibility of A_P is $G(A)/H$, where H is a minimal prime l-ideal of $G(A)$ with $G(A)^+ - H = F$. By [9], Theorem 2.1,

$$H = \{g_1 - g_2: g_i \in f(A-P)\},$$

where f is the canonical semi-valuation of K^* onto $G(A)$.

Now, since

$$\Psi(i) = \{xU(A): x \in U(R_{w_i})\} = \{xy^{-1}U(A): x, y \in A - \mathfrak{m}_i\},$$

we infer that, under this correspondence, $\mathfrak{m}_i$ corresponds to the ultrafilter $G(A)^+ - \Psi(i)$. Hence, $\Psi(i)$ is a minimal prime l-ideal of $G(A)$ and $\{G_{w_i}: i \in J\}$ is a completely regular realization of $G(A)$.

Further, let G be an l-group. Then, by [14], 7.2, there is a topology $\mathcal{T}_{\mathcal{P}}$ on G such that $(G, \mathcal{T}_{\mathcal{P}})$ is a tl-group and $\mathcal{P}$ is a base of neighbourhoods of zero. We denote by $J(A)$ the Jacobson radical of a domain A.

THEOREM 4. *Let $(K, \mathcal{T}, A)$ be a representation of $(G, \mathcal{T}_{\mathcal{P}})$. Then G is a discrete space if and only if $J(A) \neq \{0\}$.*

Proof. Let $\{\mathfrak{m}_i: i \in J\}$ be the set of maximal ideals of A. Since A is a Bézout domain, for every $i \in J$ there is a valuation w_i on K such that $A_{\mathfrak{m}_i} = R_{w_i}$. Hence, according to Lemma 4, $\{G_{w_i}: i \in J\}$ is a completely regular realization of $G(A) \cong_{\varphi} G$. Now, for every $i \in J$ there is an l-homomorphism ε_i which completes the commutative diagram

$$\begin{tikzcd}
K^* \arrow[r, "f"] \arrow[dr, "w_i"'] & G(A) \arrow[r, "\varphi"] \arrow[d, "\tau_i"] & G \arrow[dl, dashed, "\varepsilon_i"] \\
& G_{w_i} &
\end{tikzcd}$$

where f and τ_i are the canonical homomorphisms. For any $a \in A - \{0\}$ we have

$$Z(\varphi f(a)) = \{i \in J\colon \varepsilon_i \varphi f(a) = 0\} = \{i \in J\colon w_i(a) = 0\}$$
$$= \{i \in J\colon a \in A - \mathfrak{m}_i\}.$$

Assume that G is a discrete space. Then, by [14], 7.2, there is a weak unit element e in G. We may assume that $e > 0$. Hence, there exists $a \in A - \{0\}$ such that $\varphi f(a) = e$. Since $\{G_{w_i}\colon i \in J\}$ is a completely regular realization, by [11], 12.11, we have

$$Z(e) = Z(\varphi f(a)) = \emptyset.$$

Thus,

$$a \in J(A) = \bigcap \{\mathfrak{m}_i\colon i \in J\}.$$

Conversely, if $a \in J(A)$, $a \neq 0$, then $Z(\varphi f(a)) = \emptyset$ and, by [11], 12.11, and [14], 7.2, G is a discrete space.

We conclude this note by mentioning a result concerning a continuous order relation in a topological group. Recall that for a partially ordered set $(M, \leqslant)$ with a topology $\mathcal{T}$ the order relation $\leqslant$ is said to be *continuous* if for any $a, b \in M$ with $a \nleqslant b$ there are $U, V \in \mathcal{T}$ with $a \in U$, $b \in V$ such that, for every $u \in U$ and $v \in V$, $u \nleqslant v$ holds. By [17], $\leqslant$ is continuous if and only if the graph $M(\leqslant) = \{(x, y)\colon x \leqslant y\}$ of $\leqslant$ is closed in $M \times M$. The importance of this notion follows from the fact that for every tl-group with T_2-topology the order relation in G is continuous. Thus, if the group of divisibility $G(A)$ of a Bézout domain A is a tl-group with respect to the factor topology, then the division with respect to A is continuous.

We have the following simple characterization of continuous order relation in a topological order group:

PROPOSITION 3. *Let $(G, \leqslant, \mathcal{T})$ be a topological order group. Then $\leqslant$ is continuous if and only if G^+ is closed in G.*

Proof. Let G^+ be closed in G and suppose that $(x, y) \in \overline{G(\leqslant)}$ and $(x, y) \notin G(\leqslant)$, where $\overline{G(\leqslant)}$ is the closure of $G(\leqslant)$ in $G \times G$. Then we have $yx^{-1} \notin G^+$. Hence, there is a neighbourhood U of yx^{-1} such that $U \cap G^+ = \emptyset$. Now, there is a neighbourhood W (V) of x (y) such that $W^{-1} \cdot V \cap G^+ = \emptyset$. On the other hand, $W \times V$ is a neighbourhood of (x, y) in $G \times G$, and then there exists $(u, v) \in G(\leqslant) \cap W \times V$, a contradiction.

Conversely, suppose that $\leqslant$ is continuous and let $g \in \overline{G^+}$, $g \notin G^+$, where $\overline{G^+}$ is the closure of G_+ in G. Then $g^{-1} \nleqslant 1_G$ and, by [16], there is a neighbourhood U of 1_G such that $g^{-1} \nleqslant u$ for every $u \in U$. Further, since gU is a neighbourhood of g, there is $q \in G^+ \cap gU$. Thus, $1_G \leqslant q = gu$ for some $u \in U$, a contradiction.

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