COLLOQUIUM MATHEMATICUM

VOL. XXXIX 1978 FASC. 1

ON SOME MINIMAL TRANSFORMATIONS OF COMPACT SPACES

BY

WOJCIECH HYB (WARSZAWA)

The aim of the paper is to present a complete proof of a theorem on a class of minimal homeomorphisms of spaces $X \times G$, where X is a compact Hausdorff space and G is an Abelian compact group. In the case $X = G = S^1$ the theorem has been formulated in [1]; unfortunately, the proof is incomplete. Moreover, an example of orientation reversing minimal homeomorphism of the two-torus is presented.

1. Let Y be a compact Hausdorff space and let $Q\colon Y \to Y$ be a continuous transformation. A subset $M \subset Y$ is said to be *minimal* if M is closed, non-empty, invariant and for every $y \in M$ the set $\{Q^n(y)\colon n = 0, 1, 2, \ldots\}$ is dense in M. A transformation Q is said to be *minimal* if the whole space Y is minimal.

The following trivial lemma holds:

LEMMA 1. *Let Y be a compact Hausdorff space and $Q\colon Y \to Y$ a continuous transformation. If Q is minimal and f a continuous complex-valued function on Y such that $f \circ Q = f$, then f is constant.*

Let X be a compact Hausdorff space and $Q\colon X \to X$ a minimal homeomorphism. Let G be an Abelian compact group and Γ the group of all characters of G. Let $\varphi\colon X \to G$ be a continuous transformation. We define a homeomorphism $S\colon X \times G \to X \times G$ by

$$S(x, g) = \big(Q(x), g \cdot \varphi(x)\big).$$

THEOREM. *The following statements are equivalent:*

(a) *S is minimal;*

(b) *for any $\chi \in \Gamma$, $\chi \not\equiv 1$, the functional equation*

$$(*) \qquad u\big(Q(x)\big) = u(x)\chi\big(\varphi(x)\big)$$

has no continuous solution u different from 0.

Proof. (a) $\Rightarrow$ (b). Suppose that equation $(*)$ has a continuous solution for some $\chi \in \Gamma$, $\chi \not\equiv 1$. Then the function

$$f(x, g) = u(x)\frac{1}{\chi(g)}$$

s continuous and is not constant; moreover,

$$f\big(S(x, g)\big) = u\big(Q(x)\big)\frac{1}{\chi\big(g\cdot\varphi(x)\big)} = u(x)\cdot\chi\big(\varphi(x)\big)\frac{1}{\chi(g)\cdot\chi\big(\varphi(x)\big)} = f(x, g).$$

This shows that S is not minimal.

(b) $\Rightarrow$ (a). Suppose now that S is not minimal. In this case there exists a minimal set $M \neq X\times G$ (see [3]).

Let π_1: $X\times G \to X$ be the projection on the first factor. Let ψ_h: $X\times G \to X\times G$ be given by the formula

$$\psi_h(x, g) = (x, h\cdot g) \quad \text{for all } h \in G.$$

We see that $\psi_h^{-1} = \psi_{h^{-1}}$ and $S\circ\psi_h = \psi_h\circ S$ for $h \in G$. If $(x, g) \in M$, then

$$\pi_1(M) = \pi_1\big(\overline{\{S^n(x, g)\colon n = 0, 1, 2, \ldots\}}\big) = \overline{\{Q^n(x)\colon n = 0, 1, 2, \ldots\}} = X,$$

since Q is minimal and $M^x = \{g \in G\colon (x, g) \in M\} \neq \emptyset$ for all $x \in X$.

Let us choose an element $g_x \in M^x$ for any $x \in X$. We shall prove that

$$H_x = g_x^{-1}M^x = \{g_x^{-1}h\colon h \in M^x\}$$

is a subgroup of G for $x \in X$. If $g_x^{-1}h \in H_x$ and $h \in M^x$, then

$$\psi_{g_x^{-1}h}(x, g_x) = (x, h) \in M.$$

Obviously,

$$\begin{aligned}\psi_{g_x^{-1}h}(M) &= \psi_{g_x^{-1}h}\big(\overline{\{S^n(x, g_x)\colon n = 0, 1, 2, \ldots\}}\big)\\ &= \overline{\{S^n(x, h)\colon n = 0, 1, 2, \ldots\}} = M.\end{aligned}$$

It is clear that $e \in H_x$ for all $x \in X$.

If $g_x^{-1}h_i \in H_x$ $(i = 1, 2,\ h_i \in M^x)$, then

$$\psi_{g_x^{-1}h_i}(M) = M \quad \text{and} \quad \psi_{h_i^{-1}g_x}(M) = M \quad (i = 1, 2).$$

Therefore, $\psi_{h_1^{-1}g_x}(x, h_2) \in M$ and $h_1^{-1}g_xh_2 \in M^x$. Then

$$g_x(g_x^{-1}h_1)^{-1}g_x^{-1}h_2 \in M^x \quad \text{and} \quad (g_x^{-1}h_1)^{-1}g_x^{-1}h_2 \in H_x,$$

which establishes our statement.

If $g_x^{-1}h \in H_x$, then $\psi_{g_x^{-1}h}(y, g_y) \in M$. Therefore,

$$g_y g_x^{-1} h \in M^y \quad \text{and} \quad g_x^{-1} h \in H_y \quad \text{for all } x, y \in X,$$

and we conclude that $H_x = H_y = H$ for all $x, y \in X$.

It is easy to see that H is a closed subgroup and $H \neq G$.

Let $F = G/H$ be a factor group. Let π: $G \to F$ be a natural projection. Clearly, the group F is compact.

We define two continuous transformations

$$\zeta\colon X \to F \quad \text{and} \quad \mathrm{id}_X \times \pi\colon X \times G \to X \times F$$

by

$$\zeta(x) = g_x \cdot H = M^x \quad \text{and} \quad \mathrm{id}_X \times \pi(x, g) = \big(x, \pi(g)\big).$$

It is clear that $\mathrm{id}_X \times \pi$ is continuous and that

$$\pi(g_x) = g_x \cdot H = M^x, \quad \pi(g_x \cdot h) = M^x \cdot H = M^x \text{ for } h \in H.$$

We see that $A = \big\{\big(x, \zeta(x)\big)\colon x \in X\big\} = \mathrm{id}_X \times \pi(M)$ is a closed set. Therefore ζ is continuous.

If $g \in \pi^{-1}\Big(\zeta\big(Q(x)\big)\Big)$, then $\pi(g) = \zeta\big(Q(x)\big)$ and $\big(Q(x), g\big) \in M$. We see that $\Big(x, \big(\varphi(x)\big)^{-1} g\Big) \in M$ (since $S^{-1}(M) = M$) and $\pi\Big(\big(\varphi(x)\big)^{-1} g\Big) = \zeta(x)$. Therefore,

$$\pi(g) = \zeta(x) \cdot \pi\big(\varphi(x)\big)$$

and we conclude that

$$\zeta\big(Q(x)\big) = \zeta(x) \cdot \pi\big(\varphi(x)\big).$$

It is easy to see that $\mathrm{Card}\, F > 1$. In this case there exists a character $\tilde{\chi}$: $F \to S^1$ such that $\tilde{\chi} \not\equiv 1$.

Let χ: $G \to S^1$ be given by the formula $\chi = \tilde{\chi} \circ \pi$. Clearly, $\chi \in \Gamma$ and $\chi \not\equiv 1$.

Let $u = \tilde{\chi} \circ \zeta \neq 0$; we see that u is a continuous function and

$$u \circ Q(x) = \tilde{\chi}\Big(\zeta\big(Q(x)\big)\Big) = \tilde{\chi}\Big(\zeta(x) \cdot \pi\big(\varphi(x)\big)\Big) = u(x)\chi\big(\varphi(x)\big),$$

which means that u is a solution of equation (*). Thus the theorem is proved.

2. Example of orientation reversing minimal homeomorphism of the two-dimensional torus.

LEMMA 2. *Let Y be a compact connected Hausdorff space and let Q: $Y \to Y$ be a homeomorphism. Then the following statements are equivalent:*

(a) *Q is minimal;*

(b) *$Q^2 = Q \circ Q$ is minimal.*

Proof. The implication (b) $\Rightarrow$ (a) is trivial.

(a) $\Rightarrow$ (b). Suppose that Q^2 is not minimal. Then there exists a non-empty closed subset $F \subset Y$ such that $F \neq Y$ and $Q^2(F) = F$. In this case $F \cap Q(F)$ and $F \cup Q(F)$ are closed, invariant subsets of Y. If $F \cap Q(F) = \emptyset$, then $F \cup Q(F) \neq Y$ since Y is connected, and we conclude that Q is not minimal.

LEMMA 3. *Let R be the set of all real numbers and let Z be the set of all integers. Let β be an irrational number and let $b = e^{2\pi i\beta}$. Let φ: $S^1 \to S^1$ be a continuous transformation and let f: $R \to R$ be a continuous function such that*

$$\varphi(e^{2\pi ix}) = e^{2\pi i f(x)} \quad \textit{and} \quad \int_0^1 f(x)\,dx = 0.$$

Then the following statements are equivalent:

(a) *for some $k \neq 0$ the equation*

$$u(bz) = u(z)\,\varphi^k(z) \tag{1}$$

has a continuous solution $u \neq 0$;

(b) *for some $k \neq 0$ the equation*

$$r(x+\beta) - r(x) = k\cdot f(x) \tag{2}$$

has a continuous solution r such that $r(x+1) = r(x)$ for all $x \in R$.

Proof. (a) $\Rightarrow$ (b). If $u \neq 0$ is a continuous solution of (1), then $|u(z)|$ is constant since $z \mapsto bz$ is a minimal transformation of the unit circle.

Now we assume that $|u(z)| = 1$ and we may consider u as a continuous map of S^1 into S^1. In this case there exists a continuous function r: $R \to R$ such that, for some $l \in Z$,

$$r(x+1) = r(x) + l \quad \text{and} \quad u(e^{2\pi ix}) = e^{2\pi i r(x)} \quad \text{for } x \in R.$$

Since u is a solution of (1), there exists $p(x) \in Z$ such that

$$r(x+\beta) = r(x) + k\cdot f(x) + p(x) \quad \text{for all } x \in R.$$

The function p is continuous and $p(x) \in Z$, so $p(x)$ must be a constant function, say $p(x) = m_0 \in Z$. Thus

$$r(x+\beta) - r(x) = k\cdot f(x) + m_0 \quad \text{and} \quad l\cdot\beta = \int_0^1 r(x+\beta)\,dx - \int_0^1 r(x)\,dx = m_0.$$

Since β is irrational, the equality $l\beta = m_0$ implies $l = m_0 = 0$ and r is a solution of (2), $r(x+1) = r(x)$ for all $x \in R$.

(b) $\Rightarrow$ (a). If equation (2) has a continuous solution r for some $k \neq 0$, then $u(e^{2\pi ix}) = e^{2\pi i r(x)}$ is a continuous function and u is a solution of (1) for some $k \neq 0$, which completes the proof.

Now we construct the required example.

Let $(\nu_m)_{m=1}^{\infty}$ denote the sequence $\nu_1 = 2$, $\nu_{m+1} = 2^{\nu_m} - 1$. Let

$$a = \sum_{k=1}^{\infty} 2^{-2^{\nu_k}}, \quad n_m = 2^{\nu_{m+1}} \text{ for } m \geqslant 1 \quad \text{and} \quad n_{-m} = -n_m.$$

It is easy to see that

$$[n_m a] = \sum_{k=1}^{m-1} 2^{-2^{\nu_k}+\nu_{m+1}}$$

and

$$|[n_m a] - n_m a - \tfrac{1}{2}| \leqslant 2 \cdot 2^{-2^{\nu_{m+1}}+\nu_{m+1}} \leqslant 2 \cdot 2^{-n_m/2} \quad \text{for } m \geqslant 1.$$

Generally,

$$|n_m a - [n_m a] - \tfrac{1}{2}| \leqslant 2 \cdot 2^{-|n_m|/2} \quad \text{for } m \in Z \setminus \{0\}.$$

By Mean-Value Theorem applied to the function

$$h\colon \big[0, |n_m a - [n_m a] - \tfrac{1}{2}|\big] \to R^2, \quad h(x) = (\cos 2\pi x, \sin 2\pi x),$$

we obtain

$$\begin{aligned} |\exp\{2\pi i n_m a\} + 1| &= \big|h\,(|n_m a - [n_m a] - \tfrac{1}{2}|) - h(0)\big| \\ &\leqslant 2\pi\,|n_m a - [n_m a] - \tfrac{1}{2}| \leqslant 4\pi\, 2^{-|n_m|/2} \quad \text{for all } m \in Z \text{ and } m \neq 0. \end{aligned}$$

Let

$$g(x) = \sum_{m\neq 0} \frac{1}{|m|} (\exp\{2\pi i n_m a\} + 1) \exp\{2\pi i n_m x\}.$$

It is easy to see that $g\colon R \to R$ and g is a C^∞-function, since

$$\sum_{m=1}^{\infty} \frac{(2\pi n_m)^k}{2^{n_m/2}}$$

is convergent for $k = 1, 2, \ldots$

We set $\xi(e^{2\pi i x}) = e^{2\pi i g(x)}$; obviously, $\xi\colon S^1 \to S^1$ is a C^∞-transformation.

We define $S\colon T^2 \to T^2$ by

$$S(z, w) = \big(az, \overline{w}\,\xi(z)\big), \quad \text{where } a = e^{2\pi i a}.$$

We see that S is orientation reversing homeomorphism of the two-torus.

Since T^2 is connected, the map S is minimal if and only if S^2 is minimal (Lemma 2). We have

$$S^2(z, w) = \big(a^2 z, \overline{w}\,\xi(az)\overline{\xi(z)}\big).$$

Let $\varphi(z) = \xi(az)\cdot\overline{\xi(z)}$, i.e.,

$$\varphi(\exp\{2\pi i x\}) = \exp\big\{2\pi i\big(g(x+a) - g(x)\big)\big\}.$$

Let

$$f(x) = g(x+a)-g(x) = \sum_{m\neq 0} \frac{1}{|m|}(\exp\{2\pi i n_m 2a\}-1)\exp\{2\pi i n_m x\}.$$

Obviously, f is a C^∞-function,

$$\int_0^1 f(x)dx = 0 \quad \text{and} \quad f(x+1) = f(x).$$

Now we consider the following question: does there exist a continuous solution of the equation

$$r(x+2a)-r(x) = k\cdot f(x), \tag{3}$$

where $k \neq 0$ and $r(x+1) = r(x)$ for all $x \in R$?

We notice that the transformation Q: $S^1 \to S^1$, defined by $Q(z) = a^2 z$, is minimal.

Applying the Fourier expansions to (3) we get the following equation:

$$\sum_{l\in Z} b_l \exp\{2\pi i l(x+2a)\} - \sum_{l\in Z} b_l \exp\{2\pi i l x\}$$
$$= \sum_{m\neq 0} \frac{k}{|m|}(\exp\{2\pi i n_m 2a\}-1)\exp\{2\pi i n_m x\},$$

$$\sum_{l\in Z} b_l(\exp\{2\pi i l 2a\}-1)\exp\{2\pi i l x\}$$
$$= \sum_{m\neq 0} \frac{k}{|m|}(\exp\{2\pi i n_m 2a\}-1)\exp\{2\pi i n_m x\}.$$

Therefore,

$$b_l = \begin{cases} 0 & \text{if } l \neq n_m, \\ \dfrac{k}{|m|} & \text{if } l = n_m, \end{cases}$$

which implies

$$r(x) = \sum_{m\neq 0} \frac{k}{|m|}\exp\{2\pi i n_m x\}.$$

We see that this function is a unique solution of equation (3) in $L^2([0,1],\lambda)$ for $k \neq 0$ (λ is a Lebesgue measure). However, $r(x)$ cannot be equivalent to any continuous function, since

$$\sum_{m\neq 0} \frac{1}{|m|} = +\infty$$

(see [4]). Thus equation (3) has no continuous solution r such that $r(x+1) = r(x)$. Hence, by the Theorem and Lemma 3, the maps S^2 and S are minimal.

The author is indebted to Dr. W. Szlenk for suggesting the problem and to Dr. M. Misiurewicz for useful talks.

Added in proof. After the paper had been submitted for publishing we came to know that in the case where X and G are metric spaces the Theorem has been proved by W. Parry (*Compact abelian group extensions of discrete dynamical systems,* Zeitschrift für Wahrscheinlichkeitstheorie 13 (1969), p. 95-113).

REFERENCES

[1] E. G. Effros and F. Hahn, *Locally compact transformation groups and C^*-algebras*, Memoirs of the American Mathematical Society 75 (1967).

[2] H. Furstenberg, *Strict ergodicity and transformation of the torus*, American Journal of Mathematics 83 (1961), p. 573-601.

[3] W. Szlenk, *Dynamical systems*, Aarhus 1974.

[4] A. Zygmund, *Trigonometric series*, Cambridge 1968.

Reçu par la Rédaction le 15. 7. 1976