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ON BLUMBERG TYPE SPACES

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In 1921 Blumberg [2] proved that for each real-valued function f defined on the real line $\boldsymbol{R}$ there is a dense set $D \subset \boldsymbol{R}$ such that $f|D$, the restriction of f to D, is continuous. In 1960 Bradford and Goffman [3] proved that the same property holds for each metric Baire space. In fact, more general versions are known (see [1], [4], [8]).

Following Hayworth and McCoy [4] we say that a space X has *Blumberg property* with respect to a space Y if for every function $f\colon X \to Y$ there is a dense set $D \subset X$ such that $f|D$ is continuous. The aim of this note is to give some examples of spaces which have the Blumberg property with respect to each space with a countable base. We also characterize those linearly ordered topological spaces (LOTS's) with a dense order (i.e., if $a < b$, then there is a c such that $a < c < b$) which are Baire, are the union of $\leqslant 2^{\omega}$ nowhere dense sets, and have Blumberg property with respect to each space with a countable base.

1. Notation. Let f and g be families of subsets of a set X, $x \in X$, and $A \subset X$. Then

(a) $f \prec g$ (f is a *refinement* of g) iff for each $u \in f$ there is a $v \in g$ such that $u \subset v$,

(b) f is a *partition* of X iff $\bigcup f = X$ and, for each $u, v \in f$, $u = v$ or $u \cap v = \emptyset$,

(c) $f \wedge g = \{u \cap v\colon u \in f, v \in g\}$,

(d) $f|A = \{u \cap A\colon u \in f\}$,

(e) $fA = \bigcup\{u \in f\colon u \cap A \neq \emptyset\}$, $fx = f\{x\}$.

2. Main Lemma. *Let $F = \{f_n\colon n < \omega\}$ be a sequence of finite partitions of a Baire space X. Then there are a dense G_δ-set $D \subset X$ and sequences $G = \{g_n\colon n < \omega\}$ and $H = \{h_n\colon n < \omega\}$ of partitions of D such that*

(1) *$g_{n+1} \prec g_n$ and each g_n is an open partition of D,*

(2) *$h_{n+1} \prec h_n$, $h_n \prec g_n$, $h_n \prec f_n|D$ and each h_n is a family of subsets of X being of the second category at each of its points,*

(3) $\mathrm{cl}_D v = u$ *whenever* $v \in h_n$, $u \in g_n$, *and* $v \subset u$.

Conditions (1)-(3) *imply that*

(4) *there is a set* $Z \subset D$ *such that* $u \cap Z \neq \emptyset$ *for each* $u \in \bigcup G$, *and* $g_n|Z = h_n|Z$ *for each* $n < \omega$.

Proof. Without loss of generality we may assume that $f_{n+1} \succ f_n$ for each $n < \omega$. Decompose each set $w \in f_n$ into sets w' and w'' in such a way that w' is of the first category in X and w'' is of the second category at each of its points. Removing from X all the sets w', we get a dense subspace $Y \subset X$ which contains a dense G_δ-set such that each partition $f_n | Y$ consists of sets of the second category in X at each of its points. For each $w \in f_n | Y$ let us put $w^0 = \mathrm{Int}_Y \mathrm{cl}_Y w$ and $f_n^0 = \{w^0 \colon w \in f_n\}$. Each f_n^0 is a finite open covering of Y. Removing from Y all the sets of the form $\mathrm{cl}_Y w - w^0$, $w \in f_n$, $n < \omega$, we get a set $D \subset Y$ such that D contains a dense G_δ-set in X and $f_n^0 | D$ is a clopen covering of D. We may assume that D is a dense G_δ-set in X. For each $x \in D$ and $w \in f_n^0$ put

$$w(x) = \bigcap \{w \in f_n^0 \colon x \in w\} - \bigcup \{w \in f_n^0 \colon x \notin w\}.$$

The family $g_n = \{w(x) \colon x \in D, w \in f_n\}$ is a finite partition of D. Let us put $h_n = g_n \wedge f_n | D$. By the definitions, families $G = \{g_n \colon n < \omega\}$ and $H = \{h_n \colon n < \omega\}$ fulfil conditions (1) and (2). We shall show that condition (3) is also satisfied. Let $v \in h_n$ and $u \in g_n$ with $v \subset u$. There is a $w \in f_n$ such that $v = u \cap w$. The set $w \cap w^0$ is dense in w^0. By the definition of g_n and in view of $u \cap w \cap D \neq \emptyset$ we have $u \subset w^0 \cap D$. Since $w^0 \supset \mathrm{cl}_Y w$, we infer that $v = u \cap w \cap D$ is dense in u (because D is dense and G_δ in X and $w \cap D$ is of the second category at each of its points).

Now we shall show that conditions (1)-(3) imply condition (4).

Let $\bigcup G = \{u_\alpha \colon \alpha < \beta\}$. Suppose that for some $\alpha < \beta$ there have been defined sets Z_ξ for each $\xi < \alpha$ such that

(i) $Z_\gamma \subset Z_\xi$, $\gamma < \xi < \alpha$,

(ii) $u_\xi \cap Z_\xi \neq \emptyset$,

(iii) for each $x, y \in Z_\xi$, $h_n x \neq h_n y$ implies $g_n x \neq g_n y$.

Let $A = \bigcup \{Z_\xi \colon \xi < \alpha\}$. If $u_\alpha \cap A \neq \emptyset$, put $Z_\alpha = A$. If $u_\alpha \cap A = \emptyset$, then $V_m = u_\alpha - g_m A \neq \emptyset$ for some $m < \omega$. Let k be such a minimal number m. In case $k = 1$ we choose an arbitrary point $z \in V_k$ and put $Z_\alpha = A \cup \{z\}$. Then $g_n z \cap g_n x = \emptyset$ for each $x \in A$, $n < \omega$, because $\{g_n A \colon n < \omega\}$ is a decreasing family of clopen sets in D. And in case $k > 1$ there is a $y \in A$ such that $h_{k-1} y$ is dense in $g_{k-1} y$ and $V_k \cap g_{k-1} y \neq \emptyset$. Since V_k is open in D, there is a $z \in V_k \cap h_{k-1} y$. Let $Z_\alpha = A \cup \{z\}$. Taking into account the definition of the number k and using the fact that $\{g_n A \colon n < \omega\}$ is a decreasing family of sets, for each $i \geqslant k$ and $x \in A$ we have $g_i x \cap g_i z = \emptyset$. If $i < k$, we have $h_i z = h_i y$ because $h_{k-1} z = h_{k-1} y$. Hence, if $h_i x \cap h_i z = \emptyset$, then $h_i x \cap h_i y = \emptyset$ for each $x \in A$. Since x, $y \in A$, by induction we obtain $g_i x \cap g_i y = \emptyset$. Hence $g_i x \cap g_i z = \emptyset$ (because $g_i y = g_i z$ for each $i < k$). Thus condition (iii) is satisfied for $Z = A \cup \{z\}$.

Let us put $Z = \bigcup \{Z_\alpha : \alpha < \beta\}$. Condition (iii) implies that $h_n x \cap Z = g_n x \cap Z$ for each $x \in Z$, but this means that $h_n | Z = g_n | Z$.

We say that S is a *σ-disjoint pseudobase* if S is a pseudobase and $S = \bigcup \{s_n : n < \omega\}$, where each s_n is a disjoint family.

The following theorem of White [8] (see also [1] and [4]) is an easy consequence of the Main Lemma:

THEOREM 1. *If X is a Baire space which has a σ-disjoint pseudobase, then X has Blumberg property with respect to each space with a countable base.*

Proof. Let $f: X \to Y$ be a map into a space Y with a countable base $\{w_n : n < \omega\}$. Applying the Main Lemma for the sequences $F = \{f_n : n < \omega\}$, $f_n = \{f^{-1} w_n, X - f^{-1} w_n\}$, of two-element partitions of X, we get a dense G_δ-set $D \subset X$ and sequences $H = \{h_n : n < \omega\}$ and $G = \{g_n : n < \omega\}$ of partitions of D satisfying conditions (1)-(3) of the Main Lemma. Let $S = \{s_n : n < \omega\}$ be a σ-disjoint pseudobase in X. Without loss of generality it can be assumed that each s_n is a partition of a dense and open set $D_n = \bigcup s_n$. Clearly, a set $D' = D \cap \bigcap \{D_n : n < \omega\}$ is dense and G_δ in X. Moreover, sequences $g_n \wedge s_n | D$ and $h_n \wedge s_n | D'$ fulfil conditions (1)-(3) of the Main Lemma and $B = \bigcup \{g_n \wedge s_n | D' : n < \omega\}$ is a pseudobase for the space D'. From (4) of the Main Lemma it follows that there exists a set $Z \subset X$ such that $u \cap Z \neq \emptyset$ for each $u \in B$, i.e. Z is dense in X, and $g_n \wedge s_n | Z = h_n \wedge s_n | Z$, but this implies that $f^{-1} w_n \cap Z$ is open in Z. Thus $f | Z$ is a continuous map.

Now we shall use Theorem 1 for an internal characterization of some LOTS's which have Blumberg property with respect to each space with a countable base.

THEOREM 2. *Let X be a LOTS with a dense order which is a Baire space and which is the union of $\leqslant 2^\omega$ nowhere dense sets. Then X has Blumberg property with respect to each space with a countable base iff X contains a dense G_δ metrizable subspace.*

Proof. Any regular space which contains a dense metrizable subspace has a σ-disjoint pseudobase. Consequently, by Theorem 1, if X contains a dense G_δ metrizable subspace, it has Blumberg property with respect to each space with a countable base.

Now assume that X has Blumberg property with respect to each space with a countable base. In particular, X has Blumberg property with respect to the reals $\boldsymbol{R}$. Since X is the union of nowhere dense subsets, X has no isolated points. Now we can assume that X is the union of $\varkappa$, $\varkappa \leqslant 2^\omega$, pairwise disjoint nowhere dense sets, say $X = \bigcup \{F_\alpha : \alpha < \varkappa\}$. Let $\{y_\alpha : \alpha < \varkappa\}$ enumerate some distinct elements of $\boldsymbol{R}$. Define $f: X \to \boldsymbol{R}$

by letting $fx = y_a$ if $x \in F_a$. Assume that $D \subset X$ is dense and $f|D$ is continuous. Let $\{w_n: n < \omega\}$ be a countable base in $\boldsymbol{R}$. Choose, for each n, an open set $u_n \subset X$ such that $(fD)^{-1}w_n = u_n \cap D$. Since X is a LOTS, each u_n and $X - \mathrm{cl}\, u_n$ are the unions of families $\mathscr{R}_n$ and $\mathscr{R}'_n$, respectively, consisting of pairwise disjoint open intervals. We set $S_n = \mathscr{R}_n \cup \mathscr{R}'_n$. For each finite subset $t = \{n_1, \ldots, n_m\}$ of natural numbers ω we put

$$S_t = S_{n_1} \wedge S_{n_2} \wedge \ldots \wedge S_{n_m}.$$

Clearly, each S_t is a disjoint family of open intervals and $S = \{S_t: t \in [\omega]^{<\omega}\}$ is countable. Moreover, $\bigcup S_t$ is dense and open in X. Let $Y = \bigcap \{\bigcup S_t: t \in [\omega]^{<\omega}\}$. Since X is Baire, Y is dense and G_δ. We show that $S|Y$ is a base in Y. To see this, let $p \in Y$ and let $\Delta = (a, b)$ be an open interval containing p. Assume, on the contrary, that $\delta \not\subset \Delta$ for each δ such that $p \in \delta \in S_t$. This means that either $a \in \delta$ or $b \in \delta$ for each δ such that $p \in \delta \in S_t$. Assume $a \in \delta$ for each δ defined as above. Since the order in X is dense, the interval (a, p) is not empty. Hence $(a, p) \cap D \neq \emptyset$. Let us choose a point $q \in (a, p) \cap D$. Next, let us choose from $\mathscr{R}_n$ (if it is possible) an element which contains q and let $\mathscr{R}_q$ consist of all such elements. We have $\bigcap \mathscr{R}_q \subset f^{-1}fq \subset F_a$ for some a. Hence $\bigcap \mathscr{R}_q$ is nowhere dense, which implies $\bigcap \mathscr{R}_q = \{q\}$, the order in X being dense. Therefore, there are an $n < \omega$ and $\delta' \in \mathscr{R}_n$ such that $q \in \delta'$ and $p \notin \delta'$. Hence there is a $\delta \in S_n$ such that $p \in \delta$ and $q \notin \delta$. Since the members of S_n are disjoint, $\delta \cap \delta' = \emptyset$. However, in such a case, $a \in \delta$; a contradiction.

Now, since $S|Y$ is a σ-disjoint base, Y contains a dense G_δ metrizable subspace.

COROLLARY (see also [7]). *The Souslin line is not Blumberg.*

3. Remark. Repeating the second part of the Main Lemma one can obtain the following result:

Assume that any sequences $G = \{g_n: n < \omega\}$ *and* $H = \{h_n: n < \omega\}$ *of partitions of arbitrary cardinality fulfil conditions* (1)-(3) *of the Main Lemma and the condition*

(∗) *for each* $A \subset X$ *with* $|A| < \tau$ *and for each non-empty open set* $u \subset X$,

$$u - \bigcap \{g_n A: n < \omega\} \neq \emptyset.$$

Then

(5) *for each family* T *of open sets in* X *with* $|T| \leqslant \tau$ *there is a set* $Z \subset D$ *such that* $u \cap Z \neq \emptyset$ *for each* $u \in T$, *and* $g_n|Z = h_n|Z$ *for each* $n < \omega$.

The Remark implies

THEOREM 3. *Let X be a Baire space and let $\mathscr{R}$ be a countable family of open subsets of X. Then for each function $f: X \to Y$ into a space Y with a countable base there exists a subset $D \subset X$ such that $u \cap D \neq \emptyset$ for each $u \in \mathscr{R}$, and $f|D$ is continuous.*

This theorem is related to a result by Sierpiński and Zygmund [5]. They proved that under the continuum hypothesis there exists a function $f\colon \boldsymbol{R} \to \boldsymbol{R}$ such that if $f|D$ is continuous, then D is countable.

THEOREM 4. *A Baire space X is a Blumberg space whenever weight of X is $\leqslant \tau$ and the topology of X is an expansion of a metrizable topology T such that, for each $A \subset X$, $|A| < \tau$ implies that* $\mathrm{cl}_T A$ *is nowhere dense in X.*

A result analogous to this theorem (but incorrectly formulated) has been obtained by the second-named author in his doctoral dissertation [6].

THEOREM 5. *A Baire space X is a Blumberg space whenever weight of X is $\leqslant \tau$ and there is a sequence $\{s_n\colon n < \omega\}$ of open coverings of a dense G_δ-set $Y \subset X$ such that, for each $A \subset Y$, $|A| < \tau$ implies that $\bigcap\{s_n A\colon n < \omega\}$ is nowhere dense in Y.*

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