COLLOQUIUM MATHEMATICUM

VOL. XIX 1968 FASC. 1

A NOTE ON THE DENJOY INTEGRAL

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This paper continues the investigations concerning the change of variable in the Denjoy integral contained in [2] and [3]. We shall occupy ourselves with the Denjoy-Khintchine integral. The notation and terminology concerning the Denjoy integrals are the same as in [5]. We shall begin with the following theorems:

THEOREM 1. *Let f be D-integrable on an interval $[a, b]$ and φ be approximately derivable almost everywhere on an interval $[c, d]$ such that $\varphi([c, d]) \subset [a, b]$. If the function $G = F(\varphi)$, where F is an indefinite D-integral of f on $[a, b]$, is ACG on $[c, d]$, then the function $f(\varphi)\varphi'_{\mathrm{ap}}$ is D-integrable on $[c, d]$ and for $t \in (c, d]$*

$$(D) \int\limits_{\varphi(c)}^{\varphi(t)} f(x)\,dx = (D) \int\limits_{c}^{t} f\big(\varphi(t)\big)\varphi'_{\mathrm{ap}}(t)\,dt. \tag{1}$$

THEOREM 2. *Let f be D-integrable on an interval $[a, b]$ and F be an indefinite D-integral of f on $[a, b]$. Further, let φ be a continuous almost everywhere approximately derivable function fulfilling condition (N) on an interval $[c, d]$ such that*

$$\varphi([c, d]) \subset [a, b].$$

Then the following conditions are equivalent:

(i) *$G = F(\varphi)$ is ACG on $[c, d]$;*

(ii) *$f(\varphi)\varphi'_{\mathrm{ap}}$ is D-integrable on $[c, d]$ and* (1) *holds;*

(iii) *$f(\varphi)\varphi'_{\mathrm{ap}}$ is D-integrable on $[c, d]$.*

The proofs of these theorems are similar to those of Theorem 1 and Theorem 2 in [2]. It suffices to use Theorem and Lemma 3 in [4] and Theorem 3, p. 17, in [6] instead of Lemma 1 and Lemma 2 in [2].

Now we shall give the following definitions:

A function F will be said to be L (*ACM*) on a set E if F satisfies the Lipschitz condition on E (is monotone and absolutely continuous in the wide sense on E). A function F will be said to be *LG* (*ACMG*) on a set E if F is continuous on E and the set E is expressible as the sum of a finite or countable sequence of sets on each of which F is L (ACM).

The following lemmas may be proved in the standard way used in the theory of Denjoy integrals (see Theorem (9.1), p. 233, and Theorem (10.5), p. 235, in [5]):

LEMMA 1. *A continuous function is LG (ACMG) on a set E if and only if for every perfect set $E_1 \subset E$ there exists a portion P of E_1 such that F is L (ACM) on P.*

LEMMA 2. *Let F be a continuous function on a perfect set E_1. If F is not L on any portion of E_1, then the set E_2 of all points $x \in E_1$ such that*

$$|\overline{F}^{+}_{E_1}(x)| + |\underline{F}^{+}_{E_1}(x)| = +\infty \quad \textit{and} \quad |\overline{F}^{-}_{E_1}(x)| + |\underline{F}^{-}_{E_1}(x)| = +\infty$$

is dense in E_1.

The following theorem is an analogue to the one proved for the functions ACG_* in [2] (see Theorem 4 and Theorem 5):

THEOREM 3. *Let F be a function defined on an interval $[a, b]$ and let $[c, d]$ be any interval. Then the following conditions are equivalent:*

(i) *F is LG on $[a, b]$;*

(ii) *for every function φ which is ACG on $[c, d]$ and such that $\varphi([c, d]) \subset [a, b]$, the function $G = F(\varphi)$ is ACG on $[c, d]$;*

(iii) *for every function φ which is AC on $[c, d]$ and such that $\varphi([c, d]) \subset [a, b]$, the function $G = F(\varphi)$ is ACG on $[c, d]$.*

Proof. The implication (i) $\to$ (ii) easily follows from Lemma 5 in [4]. Since (ii) evidently implies (iii), it suffices to prove that (iii) $\to$ (i). To do so, let us assume that (iii) is satisfied. Suppose that, to the contrary, F is not LG on $[a, b]$. Then, since F is evidently continuous on $[a, b]$, there exists, in view of Lemma 1, a perfect set $E_1 \subset [a, b]$ such that F is not L on any portion of E_1. Let E_2 be the set from Lemma 2. Since F is continuous, it easily follows that there exist points $x_{k,n} \in E_2$, $n = 1, 2, \ldots$, $k = 1, 2, \ldots, 2^n$, positive integers $s_{k,n}$, $n = 1, 2, \ldots, k = 1, 2, \ldots, 2^{n-1}$, and positive integers l_n, $n = 1, 2, \ldots$, such that for every integer n the following conditions are satisfied:

$$x_{k,n} < x_{k+1,n} \quad \text{for} \quad k = 1, 2, \ldots, 2^n-1, \tag{2}$$

$$x_{2k,n} = x_{4k,n+1} \text{ and } x_{2k-1,n} = x_{4k-3,n+1} \quad \text{for} \quad k = 1, 2, \ldots, 2^{n-1}, \tag{3}$$

$$|F(x_{2k,n}) - F(x_{2k-1,n})| > l_n |x_{2k,n} - x_{2k-1,n}| \quad \text{for} \quad k = 1, 2, \ldots, 2^{n-1}, \tag{4}$$

$$|x_{2k,n} - x_{2k-1,n}| < \min\left(\frac{1}{3^n(s_0+1)(s_1+1)\ldots(s_{n-1}+1)}, \frac{1}{3} b_{n-1}\right) \tag{5}$$

$$\text{for} \quad k = 1, 2, \ldots, 2^{n-1},$$

$$(6) \qquad \frac{(s_0+1)(s_1+1)\dots(s_{n-1}+1)}{l_n} \leqslant \frac{1}{3^n},$$

$$(7) \qquad s_{k,n} l_n |x_{2k,n} - x_{2k-1,n}| > 1 \quad \text{for} \quad k = 1, 2, \dots, 2^{n-1},$$

$$(8) \qquad (s_{k,n}-1) l_n |x_{2k,n} - x_{2k-1,n}| \leqslant 1 \quad \text{for} \quad k = 1, 2, \dots, 2^{n-1},$$

where

$$(9) \qquad s_n = \begin{cases} \max\limits_{1\leqslant k\leqslant 2^{n-1}} (s_{k,n}) & \text{for} \quad n = 1, 2, \dots, \\ 0 & \text{for} \quad n = 0, \end{cases}$$

$$(10) \qquad b_n = \begin{cases} \min\limits_{1\leqslant k\leqslant 2^{n-1}} (x_{2k,n} - x_{2k-1,n}) & \text{for} \quad n = 1, 2, \dots, \\ 1 & \text{for} \quad n = 0. \end{cases}$$

Let us now put

$$E_3 = \bigcap_{n=1}^{\infty} \bigcup_{k=1}^{2^{n-1}} [x_{2k-1,n}, x_{2k,n}].$$

In view of (2), (3) and (5) E_3 is a perfect set of measure zero. We shall now define by induction, for $n = 1, 2, \dots$ and $k = 1, 2, \dots, 2^n$, some sets $A_C(k, n)$ and $A_O(k, n)$ consisting of ordered systems of n positive integers; with each system $(i_1, i_2, \dots, i_n)$ belonging to $A_C(k, n)$ we shall associate a closed interval $C^{k,n}_{(i_1,i_2,\dots,i_n)}$ and with each system $(i_1, i_2, \dots, i_n) \in A_O(k, n)$ an open interval $O^{k,n}_{(i_1,i_2,\dots,i_n)}$. We let first

$$A_C(2,1) = A_O(1,1) = A_O(2,1) = \{(i_1): i_1 = 1, 2, \dots, s_{1,1}\},$$

$$A_C(1,1) = \{(i_1): i_1 = 1, 2, \dots, s_{1,1}+1\}.$$

The closed intervals $C^{k,1}_{(i_1)}$ for $k = 1, 2$ and $(i_1) \in A_C(k, 1)$ and the open intervals $O^{k,1}_{(i_1)}$ for $k = 1, 2$ and $(i_1) \in A(k, 1)$ are uniquely determined by the following conditions:

$$(11) \qquad C^{1,1}_{(i_1)} < O^{1,1}_{(i_1)} < C^{2,1}_{(i_1)} < O^{2,1}_{(i_1)} < C^{1,1}_{(i_1+1)} \text{ (}^1\text{)} \quad \text{for } i_1 = 1, 2, \dots, s_{1,1},$$

$$(12) \qquad |C^{1,1}_{(i_1)}| = |O^{1,1}_{(i_1)}| = |C^{2,1}_{(i_1)}| = |O^{2,1}_{(i_1)}| = |C^{1,1}_{(i_1+1)}| = \frac{d-c}{4s_{1,1}+1}$$
$$\text{for} \quad i_1 = 1, 2, \dots, s_{1,1},$$

$$(13) \qquad \bigcup_{i_1=1}^{s_{1,1}+1} C^{1,1}_{(i_1)} \cup \bigcup_{i_1=1}^{s_{1,1}} O^{1,1}_{(i_1)} \cup \bigcup_{i_1=1}^{s_{1,1}} C^{2,1}_{(i_1)} \cup \bigcup_{i_1=1}^{s_{1,1}} O^{2,1}_{(i_1)} = [c, d].$$

Let us now suppose that for some positive integer n and $k = 1, 2, \dots, 2^n$ the sets $A_C(k, n)$, $A_O(k, n)$, the closed intervals $C^{k,n}_{(i_1,i_2,\dots,i_n)}$ for

(1) If P_1 and P_2 are intervals, then $P_1 < P_2$ means that P_1 is situated on the left of P_2.

$(i_1, i_2, \ldots, i_n) \in A_C(k, n)$ and the open intervals $O^{k,n}_{(i_1,i_2,\ldots,i_n)}$ for $(i_1, i_2, \ldots, i_n) \in A_O(k, n)$ are already defined. Then the sets $A_C(k, n+1)$, $A_O(k, n+1)$ for $k = 1, 2, \ldots, 2^{n+1}$ are defined in the following way:

$$A_C(4k-1, n+1) = A_O(4k-1, n+1) = A_O(4k, n+1)$$
$$= \{(i_1, i_2, \ldots, i_{n+1})\colon (i_1, i_2, \ldots, i_n) \in A_C(2k, n),\ i_{n+1} = 1, 2, \ldots, s_{2k,n+1}\} \quad \text{for} \quad k = 1, 2, \ldots, 2^{n-1},$$

$$A_C(4k, n+1) = \{(i_1, i_2, \ldots, i_{n+1})\colon (i_1, i_2, \ldots, i_n) \in A_C(2k, n),\ i_{n+1} = 1, 2, \ldots, s_{2k,n+1}+1\} \quad \text{for} \quad k = 1, 2, \ldots, 2^{n-1},$$

$$A_C(4k-2, n+1) = A_O(4k-2, n+1) = A_O(4k-3, n+1)$$
$$= \{(i_1, i_2, \ldots, i_{n+1})\colon (i_1, i_2, \ldots, i_n) \in A_C(2k-1, n),\ i_{n+1} = 1, 2, \ldots, s_{2k-1,n+1}\} \quad \text{for} \quad k = 1, 2, \ldots, 2^{n-1},$$

$$A_C(4k-3, n+1) = \{(i_1, i_2, \ldots, i_{n+1})\colon (i_1, i_2, \ldots, i_n) \in A_C(2k-1, n),\ i_{n+1} = 1, 2, \ldots, s_{2k-1,n+1}+1\} \quad \text{for} \quad k = 1, 2, \ldots, 2^{n-1}.$$

The closed intervals $C^{k,n+1}_{(i_1,i_2,\ldots,i_{n+1})}$ for $k = 1, 2, \ldots, 2^{n+1}$ and $(i_1, i_2, \ldots, i_{n+1}) \in A_C(k, n+1)$ and the open intervals $O^{k,n+1}_{(i_1,i_2,\ldots,i_{n+1})}$ for $k = 1, 2, \ldots, 2^{n+1}$ and $(i_1, i_2, \ldots, i_{n+1}) \in A_O(k, n+1)$ are then uniquely determined by the following conditions:

$$C^{4k,n+1}_{(i_1,i_2,\ldots,i_{n+1})} < O^{4k,n+1}_{(i_1,i_2,\ldots,i_{n+1})} < C^{4k-1,n+1}_{(i_1,i_2,\ldots,i_{n+1})} < O^{4k-1,n+1}_{(i_1,i_2,\ldots,i_{n+1})} < C^{4k,n+1}_{(i_1,i_2,\ldots,i_{n+1}+1)} \tag{14}$$

for $(i_1, i_2, \ldots, i_n) \in A_C(2k, n)$, $k = 1, 2, \ldots, 2^{n-1}$ and $i_{n+1} = 1, 2, \ldots, s_{2k,n+1}$,

$$|C^{4k,n+1}_{(i_1,i_2,\ldots,i_{n+1})}| = |O^{4k,n+1}_{(i_1,i_2,\ldots,i_{n+1})}| = |C^{4k-1,n+1}_{(i_1,i_2,\ldots,i_{n+1})}| = |O^{4k-1,n+1}_{(i_1,i_2,\ldots,i_{n+1})}| = |C^{4k,n+1}_{(i_1,i_2,\ldots,i_{n+1}+1)}| = \frac{1}{s_{2k,n+1}+1} |C^{2k,n}_{(i_1,i_2,\ldots,i_n)}| \tag{15}$$

for $(i_1, i_2, \ldots, i_n) \in A_C(2k, n)$, $k = 1, 2, \ldots, 2^{n-1}$ and $i_{n+1} = 1, 2, \ldots, s_{2k,n+1}$,

$$\bigcup_{i_{n+1}=1}^{s_{2k,n+1}+1} C^{4k,n+1}_{(i_1,i_2,\ldots,i_{n+1})} \cup \bigcup_{i_{n+1}=1}^{s_{2k,n+1}} O^{4k,n+1}_{(i_1,i_2,\ldots,i_{n+1})} \cup \bigcup_{i_{n+1}=1}^{s_{2k,n+1}} C^{4k-1,n+1}_{(i_1,i_2,\ldots,i_{n+1})} \cup \bigcup_{i_{n+1}=1}^{s_{2k,n+1}} O^{4k-1,n+1}_{(i_1,i_2,\ldots,i_{n+1})} = C^{2k,n}_{(i_1,i_2,\ldots,i_n)} \tag{16}$$

for $(i_1, i_2, \ldots, i_n) \in A_C(2k, n)$ and $k = 1, 2, \ldots, 2^{n-1}$,

$$C^{4k-3,n+1}_{(i_1,i_2,\ldots,i_{n+1})} < O^{4k-3,n+1}_{(i_1,i_2,\ldots,i_{n+1})} < C^{4k-2,n+1}_{(i_1,i_2,\ldots,i_{n+1})} < O^{4k-2,n+1}_{(i_1,i_2,\ldots,i_{n+1})} < C^{4k-3,n+1}_{(i_1,i_2,\ldots,i_{n+1}+1)} \tag{17}$$

for $(i_1, i_2, \ldots, i_n) \in A_C(2k, n)$, $k = 1, 2, \ldots, 2^{n-1}$ and $i_{n+1} = 1, 2, \ldots$ $\ldots, s_{2k-1,n+1}$,

$$(18)\qquad |C^{4k-3,n+1}_{(i_1,i_2,\ldots,i_{n+1})}| = |O^{4k-3,n+1}_{(i_1,i_2,\ldots,i_{n+1})}| = |C^{4k-2,n+1}_{(i_1,i_2,\ldots,i_{n+1})}|$$
$$= |O^{4k-2,n+1}_{(i_1,i_2,\ldots,i_{n+1})}| = |C^{4k-3,n+1}_{(i_1,i_2,\ldots,i_{n+1}+1)}| = \frac{1}{4s_{2k-1,n+1}+1}|C^{2k-1,n}_{(i_1,i_2,\ldots,i_n)}|$$

for $(i_1, i_2, \ldots, i_n) \in A_C(2k-1, n)$, $k = 1, 2, \ldots, 2^{n-1}$ and $i_{n+1} = 1, 2, \ldots$ $\ldots, s_{2k-1,n+1}$,

$$(19)\qquad \bigcup_{i_{n+1}=1}^{s_{2k-1,n+1}+1} C^{4k-3,n+1}_{(i_1,i_2,\ldots,i_{n+1})} \cup \bigcup_{i_{n+1}=1}^{s_{2k-1,n+1}} O^{4k-3,n+1}_{(i_1,i_2,\ldots,i_{n+1})} \cup$$
$$\cup \bigcup_{i_{n+1}=1}^{s_{2k-1,n+1}} C^{4k-2,n+1}_{(i_1,i_2,\ldots,i_{n+1})} \cup \bigcup_{i_{n+1}=1}^{s_{2k-1,n+1}} O^{4k-2,n+1}_{(i_1,i_2,\ldots,i_{n+1})} = C^{2k-1,n}_{(i_1,i_2,\ldots,i_n)}$$

for $(i_1, i_2, \ldots, i_n) \in A_C(2k-1, n)$ and $k = 1, 2, \ldots, 2^{n-1}$.

Define now a function φ on the set T consisting of the ends of all intervals $C^{k,n}_{(i_1,i_2,\ldots,i_n)}$, where $(i_1, i_2, \ldots, i_n) \in A_C(k, n)$, $k = 1, 2, \ldots, 2^n$ and $n = 1, 2, \ldots$, in the following way: $\varphi(t) = x_{k,n}$ if t is the end of an interval $C^{k,n}_{(i_1,i_2,\ldots,i_n)}$. In view of (3), (14), (16), (17) and (19), the function φ is well-defined. In order to prove that φ is uniformly continuous on T, let ε be any positive number. In view of (5), there exists positive integer n_0 such that $x_{2k,n_0} - x_{2k-1,n_0} < \varepsilon$ for $k = 1, 2, \ldots, 2^{n_0-1}$. Let $\delta = \min(O^{k,n}_{(i_1,i_2,\ldots,i_n)})$ where $(i_1, i_2, \ldots, i_n)$ runs over the set $A_O(k, n)$ for $k = 1, 2, \ldots, 2^n$ and $n = 1, 2, \ldots, n_0$. Now, if t_1 and t_2 are two points belonging to T such that $|t_2 - t_1| < \delta$, then $|\varphi(t_2) - \varphi(t_1)| < \varepsilon$. In fact, from the definition of $C^{k,n}_{(i_1,i_2,\ldots,i_n)}$ it follows that then there exists a positive integer $k_0 \leqslant 2^{n_0}$ such that $[t_1, t_2] \subset C^{k_0,n_0}_{(i_1^0,i_2^0,\ldots,i_n^0)}$ for some $(i_1^0, i_2^0, \ldots, i_{n_0}^0)$ $\in A_C(k_0, n_0)$. Since

$$\varphi(T \cap C^{k_0,n_0}_{(i_1^0,i_2^0,\ldots,i_{n_0}^0)}) \subset [x_{2\mathrm{E}((k_0+1)/2)-1,n_0}, x_{2\mathrm{E}((k_0+1)/2),n_0}],$$

where $\mathrm{E}(y)$ denotes the integral part of y, we have $|\varphi(t_2) - \varphi(t_1)| < \varepsilon$. Thus φ is uniformly continuous on T. Therefore φ can be extended to a function continuous on $\overline{T}$. We shall denote it also by φ.

Note that

$$(20)\qquad \overline{T} = \bigcap_{n=1}^{\infty} \bigcup_{k=1}^{2^n} \bigcup_{\substack{(i_1,i_2,\ldots,i_n)\\ \in A_C(k,n)}} C^{k,n}_{(i_1,i_2,\ldots,i_n)}$$

and in view of (25), (29) and (32) it follows that

$$|\bigcup_{k=1}^{2^n} \bigcup_{\substack{(i_1,i_2,\ldots,i_n)\\ \in A_C(k,n)}} C^{k,n}_{(i_1,i_2,\ldots,i_n)}| \leqslant \left(\frac{3}{4}\right)^n (d-c)$$

for $n = 1, 2, \ldots$ This implies that $|\overline{T}| = 0$. Now we extend φ linearly on the intervals contiguous to $\overline{T}$. This new function will also be denoted by φ. It follows from (13), (16), (19) and (20) that the intervals $O^{k,n}_{(i_1,i_2,\ldots,i_n)}$ for $(i_1, i_2, \ldots, i_n) \in A_O(k, n)$, $k = 1, 2, \ldots, 2^n$ and $n = 1, 2, \ldots$, are contiguous to $\overline{T}$. Since the set E_3 is of measure zero and $\varphi(\overline{T}) \subset E_3$, φ fulfils condition (N) on $[c, d]$. Now we shall show that φ is AC on $[c, d]$. For this purpose, in view of Theorem (7.7), p. 285 in [5], it suffices to show that φ' is integrable in the Lebesgue sense on the set of points of derivability of φ. Since $|\overline{T}| = 0$, this is equivalent to the fact that the series of increments of φ over the intervals contiguous to $\overline{T}$ is absolutely convergent. In order to prove the last condition, let us note that in view of (11), (14) and (17) we have

$$(21)\qquad \Delta(\varphi; P^{k,n}_{(i_1,i_2,\ldots,i_n)}) = x_{2\mathrm{E}((k+1)/2),n} - x_{2\mathrm{E}((k+1)/2)-1,n}\ (^2)$$

for $(i_1, i_2, \ldots, i_n) \in A_O(k, n)$, $k = 1, 2, \ldots, 2^n$ and $n = 1, 2, \ldots$ Further' from the definition of the sets $A_O(k, n)$ it follows that if $(i_1, i_2, \ldots, i_n) \in A_O(k, n)$, then

$$(22)\qquad \begin{aligned} i_j &\leqslant s_j + 1 \quad \text{for} \quad j = 1, 2, \ldots, n-1 \ (n > 1), \\ i_n &= s_{\mathrm{E}((k+1)/2),n}. \end{aligned}$$

(21) and (22) imply

$$(23)\qquad \sum_{\substack{(i_1,i_2,\ldots,i_n)\\ \in A_O(k,n)}} |\Delta(\varphi; O^{k,n}_{(i_1,i_2,\ldots,i_n)})| \leqslant s_{\mathrm{E}((k+1)/2),n}(s_0+1)(s_1+1)\ldots(s_{n-1}+1)(x_{2\mathrm{E}((k+1)/2),n} - x_{2\mathrm{E}((k+1)/2)-1,n}).$$

In view of (6) and (8) we obtain

$$(24)\qquad (s_{\mathrm{E}((k+1)/2),n}-1)(s_0+1)(s_1+1)\ldots \ldots (s_{n-1}+1)(x_{2\mathrm{E}((k+1)/2),n} - x_{2\mathrm{E}((k+1)/2)-1,n}) \leqslant 1/3^n,$$

and in view of (5) we have

$$(25)\qquad (s_0+1)(s_1+1)\ldots(s_{n-1}+1)(x_{2\mathrm{E}((k+1)/2),n} - x_{2\mathrm{E}((k+1)/2)-1,n}) < 1/3^n$$

for $k = 1, 2, \ldots, 2^n$. From (23), (24) and (25) it follows that

$$\sum_{k=1}^{2^n} \sum_{\substack{(i_1,i_2,\ldots,i_n)\\ \in A_O(k,n)}} |\Delta(\varphi; O^{k,n}_{(i_1,i_2,\ldots,i_n)})| < 2^{n+1}/3^n.$$

This completes the proof that φ is AC on $[c, d]$. Since we have assumed that (iii) is satisfied, the function $G = F(\varphi)$ ought to be ACG on $[c, d]$

(2) If φ is a function, then $\Delta(\varphi; P)$ denotes the increment of φ on the interval P.

and therefore also VBG on $[c, d]$. We shall prove that G is not VBG on $[c, d]$ and this will yield a contradiction. In view of Theorem (9.1), p. 233 in [5], it suffices to show that G is not VB on any portion of $\overline{T}$. To prove this, let us note that in view of (4) we have

$$|\varDelta(G;\, O^{k,n}_{(i_1,i_2,\ldots,i_n)})| > l_n(x_{2\mathrm{E}((k+1)/2),n} - x_{2\mathrm{E}((k+1)/2)-1,n}),$$

whence, in view of (22), we obtain

$$(26)\qquad \sum_{i_n=1}^{s_{\mathrm{E}((k+1)/2),n}} |\varDelta(G;\, O^{k,n}_{(i_1,i_2,\ldots,i_n)})| > s_{\mathrm{E}((k+1)/2),n}\, l_n(x_{2\mathrm{E}((k+1)/2),n} - x_{2\mathrm{E}((k+1)/2)-1,n})$$

for $(i_1, i_2, \ldots, i_{n-1}) \in A_C\big(\mathrm{E}((k+1)/2),\, n-1\big)$, $k = 1, 2, \ldots, 2^n$ and $n = 2, 3, \ldots$ This, in view of (7), implies

$$(27)\qquad \sum_{i_n=1}^{s_{\mathrm{E}((k+1)/2),n}} |\varDelta(G;\, O^{k,n}_{(i_1,i_2,\ldots,i_n)})| > 1$$

for $(i_1, i_2, \ldots, i_{n-1}) \in A_C\big(\mathrm{E}((k+1)/2),\, n-1\big)$, $k = 1, 2, \ldots, 2^n$ and $n = 2, 3, \ldots$ From the definition of the intervals $O^{k,n}_{(i_1,i_2,\ldots,i_n)}$ and in view of (27) it follows that the function G is not VB on any portion of the set $\overline{T}$. This completes the proof of Theorem 3.

From Theorem 3 and Theorem 2 it follows

COROLLARY 1. *Let f be D-integrable on an interval $[a, b]$, and let $[c, d]$ be any interval. Then the following conditions are equivalent:*

(i) *an indefinite D-integral of f on the interval $[a, b]$ is an LG function;*

(ii) *for every function φ which is ACG on $[c, d]$ and such that $\varphi([c, d]) \subset [a, b]$ the function $f(\varphi)\varphi'_{\mathrm{ap}}$ is D-integrable on $[c, d]$ and (1) holds;*

(iii) *for every function φ which is AC on $[c, d]$ and such that $\varphi([c, d]) \subset [a, b]$ the function $f(\varphi)\varphi'_{\mathrm{ap}}$ is D-integrable on $[c, d]$.*

The implication (i) $\to$ (ii) was proved by Tolstoff in [7] and by Karták in [1] under a stronger hypothesis.

We shall now give the characterization of ACMG-functions. For this purpose we shall prove

LEMMA 3. *If a continuous function φ fulfilling condition (N) is not ACM on any portion of a perfect set E_1, then the set E_2 of all points x belonging to E_1 such that φ is not monotone on the right at x with respect to E_1, is dense in E_1*(3).

Lemma 3 may be proved by means of Lemma 1 and the argumentation similar to that in the proof of implication (iv) $\to$ (i) in Theorem 1 in [3]. Therefore we shall omit the proof.

(3) A function φ is *monotone on the right at a point* x belonging to a set E with respect to this set if there exists a $\delta > 0$ such that either $\varphi(x) \leqslant \varphi(\overline{x})$ for $\overline{x} \in E \cap [x, x+\delta]$ or $\varphi(x) \geqslant \varphi(\overline{x})$ for $\overline{x} \in E \cap [x, x+\delta]$.

We shall now prove

THEOREM 4. *Let φ be a function defined on an interval $[c, d]$. Then the following conditions are equivalent:*

(i) *φ is ACMG on $[c, d]$;*

(ii) *for every function F which is ACMG on $\varphi([c, d])$, the function $G = F(\varphi)$ is ACG on $[c, d]$;*

(iii) *for every increasing function G which is AC on $\varphi([c, d])$ the function $G = F(\varphi)$ is ACG on $[c, d]$.*

Proof. The implication (i) → (ii) may be proved in a way similar to that used in the proof of implication (i) → (ii) in Theorem 2 of [3]. Since (ii) clearly implies (iii), it is enough to prove that (iii) → (i). For this purpose, let us assume that (iii) is satisfied. Suppose, to the contrary, that φ is not ACMG on $[c, d]$. Then, since φ is clearly continuous, there exists, in view of Lemma 1, a perfect set $E_1 \subset [c, d]$ such that φ is not ACM on any portion of E_1. Further, since φ clearly fulfils condition (N) on $[c, d]$ and therefore also on E_1, we may use Lemma 3. By this lemma there exists a sequence $\{t_n\}$ of points belonging to E_2, where E_2 is the set from the lemma, which is dense in E_1. It is easy to see that for $i = 1, 2$ and $n, k = 1, 2, \ldots$ there exist points $t^i_{k,n}$ such that

$$t^i_{k,n} \in E_1 \quad \text{for} \quad i = 1, 2 \text{ and } n, k = 1, 2, \ldots, \tag{28}$$

$$t_n < t^1_{k,n} < t^2_{k,n} \quad \text{for} \quad n, k = 1, 2, \ldots, \tag{29}$$

$$\varphi(t^1_{k,n}) < \varphi(t^1_{k+1,n}) < \varphi(t_n) < \varphi(t^2_{k+1,n}) < \varphi(t^2_{k,n}) \quad \text{for} \quad n, k = 1, 2, \ldots, \tag{30}$$

$$\lim_{k\to\infty} t^i_{k,n} = t_n \quad \text{for} \quad i = 1, 2 \text{ and } n = 1, 2, \ldots, \tag{31}$$

$$t^2_{k+1,n} < t^1_{k,n} \quad \text{for} \quad n, k = 1, 2, \ldots \tag{32}$$

Now let us define, for each positive integer n, a function F_n on the interval $[a, b] = \varphi([c, d])$ as follows:

$$F_n(x) = \begin{cases} 0 & \text{for} \quad x = \varphi(t_n), \\ 1/k & \text{for} \quad x = \varphi(t^2_{k,n}), \\ \text{otherwise piece-wise linear so that } F_n \text{ be increasing.} \end{cases}$$

Then the function F

$$F(x) = \sum_{n=1}^{\infty} \frac{F_n(x)}{n^2 M_n},$$

where $M_n = \sup\limits_{a\leqslant x\leqslant b} |F_n(x)|$, is increasing and AC on $[a, b]$ (see the proof of Theorem 3 in [3]). We shall now show that the function $G = F(\varphi)$ is not ACG on $[c, d]$ and this will contradict the hypothesis. For this purpose, it suffices to prove that G is not VB on any portion of the set E_1 (see Theorem (9.1), p. 233 in [5]). To prove this, let us suppose that P is any portion of E_1. Then, since the sequence $\{t_n\}$ is dense in E_1, there exists a positive integer n_0 such that $t_{n_0} \epsilon P$. In view of (31) there exists a positive integer k_{n_0} such that for $k \geqslant k_{n_0}$ the points t^1_{k,n_0}, t^2_{k,n_0} belong to P. From the definition of F and in view of (30) we obtain

$$|G(t^2_{k,n_0}) - G(t^1_{k,n_0})| > \frac{1}{kn_0^2 M_{n_0}}$$

for each positive integer k. Since, in view of (32), the intervals $[t^1_{k,n_0}, t^2_{k,n_0}]$ are non-overlapping and their ends belong to P, G is not VB on P. This completes the proof of Theorem 4.

From Theorem 4 and Theorem 2 follows

COROLLARY 2. *Let φ be a continuous, almost everywhere approximately derivable function fulfilling condition* (N) *on an interval* $[c, d]$. *Then the following conditions are equivalent*:

(i) *φ is ACMG on* $[c, d]$;

(ii) *for every function f D-integrable on* $\varphi([c, d])$, *the function* $f(\varphi)\varphi'_{\mathrm{ap}}$ *is D-integrable on* $[c, d]$ *and* (1) *holds*;

(iii) *for every non-negative function f integrable on* $\varphi([c, d])$ *in the Lebesgue sense, the function* $f(\varphi)\varphi'_{\mathrm{ap}}$ *is D-integrable on* $[c, d]$.

The implication (i) $\to$ (ii) was proved by Tolstoff in [8] under a stronger hypothesis on function φ.

Since the following theorem can be proved in the standard way used in the theory of the Denjoy integrals (Romanowski's lemma), we shall omit the proof.

THEOREM 5. *Let φ be a continuous, almost everywhere approximately derivable function fulfilling condition* (N) *on an interval* $[c, d]$. *Further, let f be a function defined on* $\varphi([c, d])$ *such that the function* $f(\varphi)\varphi'_{\mathrm{ap}}$ *is D-integrable* (D_**-integrable*) *on* $[c, d]$. *Then the function f is D-integrable* (D_**-integrable*) *on* $\varphi([c, d])$.

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Reçu par la Rédaction le 30.12.1966