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LATTICE ORDERED GROUPS WITH COMPLETE EPIMORPHIC IMAGES

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Let $\mathscr{A}$ be a class of universal algebras of the same type (i.e., each algebra of this class has the same set of operations). We denote by $E(\mathscr{A})$ the class of all algebras A with the property that each epimorphic image of A belongs to $\mathscr{A}$. The natural question arises to characterize the class $E(\mathscr{A})$ for a given $\mathscr{A}$. Let $\mathscr{A}$ be the class of all archimedean lattice ordered groups; l-groups $G \in E(\mathscr{A})$ (called ***hyper-archimedean*** or ***epi-archimedean***) were investigated in [1], [2], [4] and [6]. Birkhoff (see [3], Problem 32) proposed the following problem: describe the class $E(\mathscr{L})$, where $\mathscr{L}$ is the class of all complete lattices.

In this note we shall characterize the class $E(\mathscr{G})$, where $\mathscr{G}$ is the class of all complete lattice ordered groups. We show that an l-group belongs to $E(\mathscr{G})$ if and only if it is a restricted direct product of linearly ordered groups G_i such that each G_i is isomorphic either to the additive group of all reals or to the additive group of all integers. Each closed l-subgroup H of an l-group $G \in E(\mathscr{G})$ belongs to $E(\mathscr{G})$. On the other hand, we show that each complete lattice L can be embedded into a lattice L_1 belonging to $E(\mathscr{L})$ and such that L is a closed sublattice of L_1. From this it follows that the class $E(\mathscr{L})$ cannot be characterized by identities involving a finite or an infinite number of variables.

1. Complete lattice ordered groups. For the terminology and notations concerning lattices and lattice ordered groups, cf. Birkhoff [3] and Fuchs [7]. A lattice ordered group G is called ***complete*** if each bounded non-empty subset of G has the supremum.

Let G_1 and G_2 be lattice ordered groups. Assume that there exists a homomorphism φ of G_1 onto G_2 (i.e., G_2 is an epimorphic image of G_1). The homomorphism φ is called ***complete*** if it satisfies the following condition: if $\{x_i\} \subset G_1$ and $\bigvee x_i$ exists in G_1, then $\bigvee \varphi(x_i)$ exists in G_2 and $\varphi(\bigvee x_i) = \bigvee \varphi(x_i)$.

A system $\emptyset \neq X \subset G_1$ is said to be *disjoint* if $x_1 \wedge x_2 = 0$ for any pair of distinct elements of the set X and $x \geqslant 0$ for each $x \in X$.

Let G be a complete lattice ordered group. Assume that $X = \{x_i\}$ $(i \in I)$ is a disjoint subset of G such that each element of X is strictly positive, $\operatorname{card} X \geqslant \aleph_0$, and the set X is bounded in G. Let M be the set of all elements $y \in G^+$ such that

$$y = \bigvee_{i \in I_1} x_i \quad \text{for some } I_1 \subset I;$$

if $I_1 = \emptyset$, we put $y = 0$.

LEMMA 1. *The set M is a closed sublattice of G and M is an atomic Boolean algebra.*

Proof. Let

$$y = \bigvee_{i \in I_1} x_i, \quad z = \bigvee_{j \in I_2} x_j, \quad I_1, I_2 \subset I.$$

Then

$$y \vee z = \bigvee_{i \in I_1 \cup I_2} x_i,$$

and, since the set X is disjoint,

$$y \wedge z = \bigvee_{i \in I_1} \bigvee_{j \in I_2} (x_i \wedge x_j) = \bigvee_{i \in I_1 \cap I_2} x_i.$$

Thus M is a sublattice of G. Write

$$x = \bigvee_{i \in I} x_i.$$

Elements x and 0 are the greatest and the least elements of M, respectively. The lattice M is distributive, because G is distributive. Put

$$y^* = \bigvee_{i \in I \setminus I_1} x_i.$$

Then we have $y \vee y^* = x$ and $y \wedge y^* = 0$, and so y^* is the complement of y in M. Therefore, M is a Boolean algebra. Obviously, X is the set of all atoms of M, and so M is atomic. It remains to verify that M is a closed sublattice of G.

Let

$$\{y_k\}_{k \in K} \subset M, \quad y_k = \bigvee_{i \in I_k} x_i, \quad I_k \subset I.$$

Then

$$\bigvee_{k \in K} y_k = \bigvee_{i \in \cup I_k} x_i.$$

Put

$$y_0 = \bigvee_{i \in \cap I_k} x_i.$$

Clearly, $y_0 \leqslant y_k$ for each $k \in K$. Let $z \in G^+$, and $z \leqslant y_k$ for each $k \in K$. Since $z \wedge y_k^* = 0$ for each $k \in K$, we obtain

$$z \wedge \Big(\bigvee_{k \in K} y_k^*\Big) = 0.$$

We have

$$y_1 = \bigvee_{k\in K} y_k^* = \bigvee_{k\in K} \bigvee_{i\in I\setminus I_k} x_i = \bigvee_{i\in \cup(I\setminus I_k)} x_i = \bigvee_{i\in I\setminus \cap I_k} x_i.$$

Thus $y_1 = y_0^*$. Since

$$z = z\wedge x = z\wedge (y_0 \vee y_1) = (z\wedge y_0)\vee(z\wedge y_1) = z\wedge y_0,$$

we obtain $z \leqslant y_0$. From this it follows that y_0 is the least upper bound of the set $\{y_k\}_{k\in K}$ in G. Therefore, M is a closed sublattice of G.

Under the same notation as above let A be the l-ideal of the l-group G generated by the set X and let B be the ideal of the Boolean algebra M generated by the set X. To the l-ideal A (or ideal B) there corresponds a partition $\varrho(A)$ (or $\varrho(B)$) of G (or M). We write $x\equiv y(A)$ if the elements $x, y\in G$ belong to the same class of $\varrho(A)$; the notation $x \equiv y(B)$ for $x, y\in M$ has an analogous meaning.

Lemma 2. *Let A_1 be the set of all elements $s\in G$ such that there exist elements $x_1, \ldots, x_k\in X$ and positive integers $n_1, \ldots, n_k$ satisfying*

$$-(n_1x_1+\ldots+n_kx_k) \leqslant s \leqslant n_1x_1+\ldots+n_kx_k.$$

Then $A_1 = A$.

Proof. It is easy to verify that A_1 is a convex l-subgroup of G generated by the set X. Since G is complete, it is commutative, and so A_1 is an l-ideal of G. Therefore, $A_1 = A$.

Lemma 3. *Let B_1 be a complete atomic Boolean algebra,* $\operatorname{card} B_1 \geqslant \aleph_0$, *and let B be the ideal of B_1 generated by the set of all atoms of B. Then the Boolean algebra B_1/B is not complete.*

This is an easy consequence of Theorem 21.4 of [10].

Lemma 4. *Let G, M, A and B be given as above, and let $p, q\in M$. Then $p \equiv q(B)$ if and only if $p \equiv q(A)$.*

Proof. Let $p \equiv q(B)$. Since the partition $\varrho(B)$ corresponds to the ideal B of M, there exist elements $b_1, b_2\in B$ such that

(1) $$p\vee b_1 = q\vee b_2.$$

Obviously, $b_1, b_2\in A$. Since $\varrho(A)$ is a congruence with respect to the operations $\wedge$, $\vee$ and $+$, it follows from (1) that the elements p and q belong to the same class of the partition $\varrho(A)$.

Assume that $p \equiv q(A)$, $p \neq q$. Write $p\wedge q = p_1$ and $p\vee q = q_1$, and let s be the relative complement of p_1 in the interval $[0, q_1]$. Then $s\in M$. Since $\varrho(A)$ is a congruence with respect to the lattice operations, we obtain $p_1 \equiv q_1(A)$, and since the intervals $[p_1, q_1]$ and $[0, s]$ are transposed to each other, we have $0 \equiv s(A)$. According to Lemma 2, there are elements $x_1, \ldots, x_k\in X$ and positive integers $n_1, \ldots, n_k$ such that

$$0 \leqslant s \leqslant n_1x_1+\ldots+n_kx_k.$$

From this it follows that $s \wedge x_i = 0$ for each $x_i \in X \setminus \{x_1, \ldots, x_k\}$. Since $s \in M$, the element s is the join of some elements of X. Therefore,

$$s = \bigvee_{i \in I_1} x_i, \quad \{x_i\}_{j \in I_1} \subset \{x_1, \ldots, x_k\}.$$

Hence the set I_1 is finite and this implies $s \in B$. Thus $0 \equiv s(B)$ and from this it follows that $p \equiv q(B)$.

Let G be an l-group, $\emptyset \neq Z \subset G$. We write

$$Z^\delta = \{g \in G\colon |g| \wedge |z| = 0 \text{ for each } z \in Z\}.$$

The set Z^δ is a closed convex l-subgroup of G (Šik [11]). If G is a complete l-group, then Z^δ is a direct factor of G (see [3], Chapter XIV). For $Z = \{x\}$, we write $Z^{\delta\delta} = [x]$. The component of an element t of a complete l-group G in the direct factor $[x]$ will be denoted by $t[x]$. For $0 \leqslant t \in G$ and $0 \leqslant x$, we have

$$t[x] = \sup\{z \in [x]\colon z \leqslant t\}.$$

If $t = z_1 \vee z_2$, $z_1 \in [x]$ and $z_2 \wedge x = 0$, then $t[x] = z_1$.

We use the same notation as above. For $y \in G$ and $z \in M$ we denote by $\tilde{y}$ and $\bar{z}$ the classes of the partitions $\varrho(A)$ and $\varrho(B)$ containing the elements y and z, respectively. If

$$z_1, z_2 \in M, \quad z_1 = \bigvee_{i \in I_1} x_i, \quad z_2 = \bigvee_{i \in I_2} x_i, \quad I_1, I_2 \subset I,$$

then $\bar{z}_1 \leqslant \bar{z}_2$ if and only if the set $I_3 = I_1 \setminus I_2$ is finite. Put

$$z_{10} = \bigvee_{i \in I_3} x_i, \quad z_{11} = \bigvee_{i \in I \setminus I_3} x_i.$$

Assume that $\bar{z}_1 \leqslant \bar{z}_2$. Then $z_{10} \in B \subset A$, and since $\varrho(A)$ is a congruence relation with respect to the operation $\vee$, we obtain $\tilde{z}_1 \leqslant \tilde{z}_2$. From this and from Lemma 4 we infer that $\bar{z}_1 < \bar{z}_2$ implies $\tilde{z}_1 < \tilde{z}_2$.

LEMMA 5. *Let G be a complete lattice ordered group containing an infinite disjoint subset X. Let A be the l-ideal of G generated by the set X. Then the factor l-group G/A is not complete.*

Proof. For any $y \in [0, x]$, we have

$$y = y \wedge x = y \wedge \Big(\bigvee_{i \in I} x_i\Big) = \bigvee_{i \in I} (y \wedge x_i),$$

and since the set $X = \{x_i\}_{i \in I}$ is disjoint, $y[x_i] = y \wedge x_i$ for each $i \in I$. If $y \in M$, i.e., if

$$y = \bigvee_{i \in I_1} x_i \quad \text{for some } I_1 \subset I,$$

then $y[x_i] = x_i$ for $i \in I_1$, and $y[x_i] = 0$ for $i \in I \setminus I_1$. Let $y, z \in [0, x]$.

From $y \leqslant z$ it follows that $y[x_i] \leqslant z[x_i]$ for each $i \in I$. Conversely, if $y[x_i] \leqslant z[x_i]$ for each $i \in I$, then

$$y = \bigvee_{i \in I} (y \wedge x_i) \leqslant \bigvee_{i \in I} (z \wedge x_i) = z,$$

whence $y \leqslant z$.

From Lemmas 1 and 3 it follows that there exists a subset $\emptyset \neq Y = \{y_k\}_{k \in K} \subset M$ such that the set $\overline{Y} = \{\bar{y}_k\}_{k \in K}$ has no least upper bound in M/B. Let us consider the set $\tilde{Y} = \{\tilde{y}_k\}_{k \in K}$ and let $v \in G$, $v \leqslant x$, $\tilde{y}_k \leqslant \tilde{v}$ for each $k \in K$. For $i \in I$, we put $z_i = x_i$ if $v[x_i] = x_i$, and $z_i = 0$ otherwise. Write $z = \bigvee z_i$ $(i \in I)$. Then $z \leqslant v$ and $z \in M$. Let $k \in K$ be fixed, and

$$y_k = \bigvee_{i \in I_k} x_i.$$

Write $(y_k - v) \vee 0 = t$. From $\tilde{y}_k \leqslant \tilde{v}$ we obtain $\tilde{t} = \tilde{0}$, whence $t \in A$. There exist distinct elements $x_1, \ldots, x_m \in X$ and positive integers $n_1, \ldots, n_m$ such that

$$0 \leqslant t \leqslant n_1 x_1 + \ldots + n_m x_m.$$

Let $x_i \in X \setminus \{x_1, \ldots, x_m\} = X_1$. We have $t[x_i] = 0$; thus

$$(y_k[x_i] - v[x_i]) \vee 0 = 0,$$

and so $y_k[x_i] \leqslant v[x_i]$. From $v \leqslant x$ we infer that $v[x_i] \leqslant x_i$. If $y_k[x_i] = 0$, then $y_k[x_i] \leqslant z[x_i]$. If $y_k[x_i] \leqslant x_i$, then $v[x_i] = x_i$, whence $z[x_i] = y_k[x_i]$. Therefore, $y_k[x_i] \leqslant z[x_i]$ for each $x_i \in X \setminus \{x_1, \ldots, x_m\}$. Write

$$y_k^0 = \bigvee y_k[x_i] \ (x_i \in X_1), \qquad y_k^1 = \bigvee y_k[x_i] \ (x_i \in X \setminus X_1).$$

Then $y_k^1 \in B$, $y_k^0 \in M$ and $y_k = y_k^0 \vee y_k^1$, $y_k^0 \leqslant z$. Hence $\tilde{y}_k \leqslant \tilde{z}$ for each $k \in K$. At the same time we have $\bar{y}_k \leqslant \bar{z}$ for each $k \in K$. Since $\overline{Y}_k$ has no supremum in M, there exists a $u \in M$ such that $\bar{y}_k \leqslant \bar{u}$ for each $k \in K$ and $\bar{u} < \bar{z}$. From this it follows that $\tilde{y}_k \leqslant \tilde{u}$ for each $k \in K$ and $\tilde{u} < \tilde{z} \leqslant \tilde{v}$. This proves that the set Y has no supremum in G/A.

We denote by Z^+ (R^+) the additive l-group of all integers (all reals) with the natural linear order.

LEMMA 6. *Let G be a complete lattice ordered group such that each bounded disjoint subset of G is finite. Then each epimorphic image of G is complete.*

Proof. From Theorem 6.1 of [5] and from the fact that G is complete it follows that G is a restricted direct product of linearly ordered groups A_i $(i \in I)$. Since each A_i is complete, it is isomorphic either to R^+ or to Z^+. Let H be an l-ideal of G and let $I_1 = \{i \in I : A_i \subset H\}$. Then G/H is isomorphic to the restricted direct product of l-groups A_i $(i \in I \setminus I_1)$. Therefore, G/H is a complete l-group.

COROLLARY. *If H is a closed l-subgroup of an l-group G belonging to $E(\mathscr{G})$, then H belongs to $E(\mathscr{G})$.*

In fact, H is a complete l-group and each bounded disjoint subset of H is finite; hence $H \in E(\mathscr{G})$.

LEMMA 7. *Let φ be a homomorphism of a complete l-group G onto an l-group H. Then φ is complete if and only if $\varphi^{-1}(0)$ is a closed l-subgroup of G.*

Proof. Assume that φ is complete, $\{g_i\} \subset \varphi^{-1}(0)$ $(i \in I)$, and $\bigvee g_i = g$. Then $\varphi(\bigvee g_i) = \bigvee \varphi(g_i) = 0$, whence $g \in \varphi^{-1}(0)$. Conversely, assume that $\varphi^{-1}(0)$ is a closed l-subgroup of G and let $g_i \in G$, $\bigvee g_i = g$. Obviously, $\varphi(g_i) \leqslant \varphi(g)$ for each $i \in I$. Suppose that there is a $z \in G$ such that $\varphi(g_i) \leqslant \varphi(z) \leqslant \varphi(g)$ for each $i \in I$. Put $z' = z \wedge g$. We have $\varphi(g_i) \leqslant \varphi(z') \leqslant \varphi(g)$. Write $z_i = g_i \wedge z'$. We obtain $\varphi(z_i) = \varphi(g_i) \wedge \varphi(z') = \varphi(g_i)$, whence $\varphi(g_i - z_i) = 0$ for each $i \in I$. Further, $0 \leqslant g_i - z_i \leqslant g_i \leqslant g$; thus there exists a $z_0 = \bigvee (g_i - z_i) \geqslant 0$. Since $g_i - z_i \in \varphi^{-1}(0)$, by the assumption we have $z_0 \in \varphi^{-1}(0)$. Then $z_i \leqslant z'$ and

$$z_0 + z' = \bigvee (g_i - z_i) + z' = \bigvee (g_i - z_i + z) \geqslant \bigvee g_i = g.$$

From this we obtain $\varphi(z_0) + \varphi(z') \geqslant \varphi(g)$. Since $\varphi(z_0) \geqslant 0$, we have $\varphi(z') \geqslant \varphi(g)$, and hence $\varphi(z') = \varphi(g)$. Therefore, we have $\varphi(z) = \varphi(g)$ and $\bigvee \varphi(g_i) = \varphi(g)$.

LEMMA 8. *Let G be given as in Lemma 6 and let φ be a homomorphism of G onto an l-group G_1. Then the homomorphism φ is complete.*

Proof. Let $A = \varphi^{-1}(0)$ and let I_1 be given as in the proof of Lemma 6. Then A is the restricted subdirect product of l-ideals A_i $(i \in I_1)$, and so A is a direct factor of G. Thus A is a closed l-ideal of G. From this and from Lemma 7 it follows that φ is a complete homomorphism.

If G is a complete l-group and if φ is a complete homomorphism of G onto an l-group G_1, then, clearly, G_1 is complete. Thus from Lemmas 5-8 we obtain

THEOREM 1. *Let G be an l-group. Then the following conditions are equivalent:*

(i) *Each epimorphic image of G is complete.*

(ii) *G is a restricted direct product of linearly ordered groups A_i $(i \in I)$ such that, for each $i \in I$, A_i is isomorphic to R^+ or Z^+.*

(iii) *G is complete and each homomorphism on G is complete.*

COROLLARY. *Let G be a lattice ordered group such that each epimorphic image of G is complete. Then G is hyper-archimedean.*

This follows from Theorem 1 by the use of condition (v) from [4], p. 363.

2. Complete lattices. Let L be a complete lattice such that $\operatorname{card} L > 1$. For each pair $x, y \in L$ with $x < y$, we construct four new elements $u_1(x, y)$, $u_2(x, y)$, $v_1(x, y)$ and $v_2(x, y)$, and the set of these elements we denote by $A(x, y)$. Let $L_1 = L \cup \left(\bigcup A(x, y)\right)$ with $x, y \in L$ and $x < y$. We denote by 0 and 1 the least and the greatest element of L, respectively. Consider the following partial order in L_1:

(i) For $x, y \in L$, we put $x \leqslant y$ in L_1 if and only if $x \leqslant y$ in L.

(ii) For $x, y \in L$, $x < y$, $z \in L$, we put $u_i(x, y) \geqslant z$ if and only if $x \geqslant z$, and $u_i(x, y) \leqslant z$ if and only if $z = 1$ $(i = 1, 2)$.

(iii) For $x, y \in L$, $x < y$, $z \in L$, we put $v_i(x, y) \leqslant z$ if and only if $y \leqslant z$, and $v_i(x, y) \geqslant z$ if and only if $z = 0$ $(i = 1, 2)$.

(iv) For $z_1, z \in L_1 \setminus L$, we put $z_1 \leqslant z_2$ if either $z_1 = z_2$ or there is an $x \in L$ such that $z_1 < x < z_2$.

LEMMA 9. *The set L_1 with the relation $\leqslant$ is a complete lattice.*

Proof. Let us write $U = \{u_1(x, y), u_2(x, y)\}$ $(x, y \in L, x < y)$, and $V = \{v_1(x, y), v_2(x, y)\}$ $(x, y \in L, x < y)$. Let $\emptyset \neq M \subset L_1$. Let Y be the set of all $y \in L$ such that $v_i(x, y) \in V \cap M$ for some $x \in L$ and some $i \in \{1, 2\}$. We distinguish two cases.

(i) $M \cap U \neq \emptyset$, $u^1 \in M \cap U$.

For each $u \in U$ and each $z \in L_1$, we have either $z \leqslant u$ or $\sup_{L_1}\{u, z\} = 1$. Hence either $\sup_{L_1} M = u^1$ or $\sup_{L_1} M = 1$.

(ii) $M \cap U = \emptyset$.

If $v_i(x_1, y_1)$ and $v_j(x_2, y_2)$ $(i, j \in \{1, 2\})$ are distinct elements of V, $z \in L$, then

$$\sup_{L_1}\{v_i(x_1, y_1),\ v_j(x_2, y_2)\} = y_1 \vee y_2, \qquad \sup_{L_1}\{v_i(x_1, y_1),\ z\} = y_1 \vee z.$$

From this it follows that

$$\sup_{L_1} M = \sup_{L} Y \quad \text{whenever card } M > 1.$$

For the infimum we can apply a dual method. Thus L_1 is complete and L is a closed sublattice of L_1.

Let ϱ be a congruence relation on a lattice L, $x, y \in L$, and $x \leqslant y$. We say that the interval $[x, y]$ is *anulled* in ϱ if $x \equiv y\ (\varrho)$.

LEMMA 10. *The lattice L_1 is simple (i.e., it has no non-trivial congruence relation).*

Proof. Let ϱ be a congruence relation on L_1 and assume that $p \equiv q\ (\varrho)$ for some $p, q \in L_1$, $p \neq q$. Then $p \wedge q = p \vee q\ (\varrho)$. Each non-trivial interval of L_1 contains one of the subintervals

$$(2) \quad [u_i(x, y), 1], \quad [x, u_i(x, y)], \quad [x, y], \quad [v_i(x, y), y], \quad [0, v_i(x, y)],$$

where $i \in \{1, 2\}$, $x, y \in L$, $x \neq y$. Any two of intervals (2) are projective, and hence if any of them is anulled in ϱ, then all are anulled in ϱ; therefore, $0 \equiv 1\ (\varrho)$.

From Lemma 10 it follows that if L' is a homomorphic image of L_1, then either L_1 is isomorphic to L' or $\operatorname{card} L' = 1$. Hence $L_1 \in E(\mathscr{L})$. Thus we have

THEOREM 2. *Each complete lattice L is a closed sublattice of a lattice L_1 belonging to $E(\mathscr{L})$.*

If a complete lattice fulfils some identity (with a finite or an infinite number of variables), then each of its closed sublattices fulfils this identity. Let us consider sentences containing symbols $=$, $\leqslant$, $\wedge$, $\vee$ and variables whose range is the set of elements of a lattice, together with the logical symbols for conjunction, disjunction and quantification. Such sentences will be called *positive*. Positive sentences for abstract algebraic systems were considered by Marczewski [9] and Lyndon [8]. A positive sentence that does not contain the symbol for existential quantification will be called *strictly positive*. If a strictly positive sentence is valid for a lattice L, then it is valid for each sublattice of L. (Let us remark that the analogous assertion does not hold for positive sentences.) Therefore, from Theorem 2 we obtain

COROLLARY. *The class $E(\mathscr{L})$ cannot be defined by strictly positive properties. The class $E(\mathscr{L})$ cannot be defined by identities involving a finite or an infinite number of variables.*

REFERENCES

[1] K. A. Baker, *Topological methods in the algebraic theory of vector lattices*, Dissertation, Harvard University, 1964.

[2] A. Bigard, *Groupes archimédiens et hyper-archimédiens*, Séminaire Dubreil-Pisot (2) 21 (1967-1968).

[3] G. Birkhoff, *Lattice theory*, third edition, Providence 1967.

[4] R. D. Byrd, P. Conrad and J. T. Lloyd, *Characteristic subgroups of lattice ordered groups*, Transactions of the American Mathematical Society 158 (1971), p. 339-371.

[5] P. Conrad, *Some structure theorems for lattice ordered groups*, ibidem 99 (1961), p. 1-29.

[6] — *Epi-archimedean groups* (preprint).

[7] Л. Фукс, *Частично упорядоченные алгебраические системы*, Москва 1965.

[8] R. C. Lyndon, *Properties preserved under homomorphism*, Pacific Journal of Mathematics 9 (1959), p. 143-154.

[9] E. Marczewski, *Sur les congruences et les propriétés positives d'algèbres abstraites*, Colloquium Mathematicum 2 (1951), p. 220-228.

[10] R. Sikorski, *Boolean algebras*, Berlin 1964.

[11] Ф. Шик, *К теории структурно упорядоченных групп*, Чехословацкий математический журнал 6 (1956), p. 1-25.

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