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EXISTENCE OF BOREL LIFTING

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It is the purpose of this note to give a short proof of the theorem of von Neumann and Stone [2] on the existence of a Borel lifting in the case of a separable metric space with a measure. The continuum hypothesis is assumed.

Let $\mathscr{A}$ be a σ-ring of subsets of a set $X \neq \emptyset$ and let $\mathscr{I} \subset \mathscr{A}$ be a σ-ideal. $A \sim B$ means that $A \Delta B \in \mathscr{I}$, $A^0 = X - A$, and $a(\mathscr{F})$ is the smallest algebra of sets containing the family $\mathscr{F} \subset 2^X$.

Definition. Let $\mathscr{D} \subset \mathscr{A}$ be an arbitrary ring (algebra). A mapping $\varrho\colon \mathscr{D} \to \mathscr{A}$ is called a *$(\mathscr{D}, \mathscr{A})$-lifting with respect to $\mathscr{I}$* if it has the following properties:

(i) $\varrho(A) \sim A$,

(ii) if $A \sim B$, then $\varrho(A) = \varrho(B)$,

(iii) $\varrho(\emptyset) = \emptyset$ (and $\varrho(X) = X$),

(iv) $\varrho(A \cap B) = \varrho(A) \cap \varrho(B)$,

(v) $\varrho(A \cup B) = \varrho(A) \cup \varrho(B)$.

An $(\mathscr{A}, \mathscr{A})$-lifting with respect to $\mathscr{I}$ will be called a *lifting of $\mathscr{A}$ with respect to $\mathscr{I}$*.

LEMMA. *Let $\mathscr{D} \subset \mathscr{A}$ be at most countable algebra and let $H \in \mathscr{A} - \mathscr{D}$ be an arbitrary set. For every $(\mathscr{D}, \mathscr{A})$-lifting $\bar{\varrho}$ with respect to $\mathscr{I}$, there exists an $\big(a(\mathscr{D} \cup \{H\}), \mathscr{A}\big)$-lifting ϱ with respect to $\mathscr{I}$ which is an extension of $\bar{\varrho}$.*

Proof. For every $M \in \mathscr{A}$, let $\mathscr{T}_1(M)$ be the family of all sets $A \in \mathscr{D}$ such that $A \cap M \sim \emptyset$, and let $\mathscr{T}_2(M)$ be the σ-ring of all sets $A \in \mathscr{A}$ with the same property. It is obvious that $A \in \mathscr{T}_1(M)$ implies $\bar{\varrho}(A) \in \mathscr{T}_2(M)$.

Let us put $M_\infty = \bigcup \{\bar{\varrho}(A)\colon A \in \mathscr{T}_1(M)\}$. Of course, $M_\infty \in \mathscr{T}_2(M)$.

Now we shall prove that $M_\infty \cap (M^0)_\infty = \emptyset$. Suppose it is false; then there exist $A \in \mathscr{T}_1(M)$ and $B \in \mathscr{T}_1(M^0)$ such that $\bar{\varrho}(A) \cap \bar{\varrho}(B) \neq \emptyset$. Hence $A \cap B \nsim \emptyset$. On the other hand, $A \cap M \sim \emptyset$ and $B \cap M^0 \sim \emptyset$, so that we have $A \cap B \sim \emptyset$.

Let us put

$$\varrho(H) = H \cap (H_\infty)^0 \cup H^0 \cap (H^0)_\infty$$

and

$$\varrho(H^0) = H^0 \cap [(H^0)_\infty]^0 \cup H \cap H_\infty.$$

It follows that $\varrho(H^0) = X - \varrho(H)$.

It is also easy to see that, for every $C \in \mathcal{T}_1(H)$, we have $\bar{\varrho}(C) \cap \varrho(H) = \emptyset$ and, for every $C \in \mathcal{T}_1(H^0)$, $\bar{\varrho}(C) \cap \varrho(H^0) = \emptyset$.

If $E \in a(\mathcal{D} \cup \{H\})$ is an arbitrary set, then there exist A and B from $\mathcal{D}$ such that $E = (A \cap H) \cup (B \cap H^0)$.

Let us put

$$\varrho(E) = [\bar{\varrho}(A) \cap \varrho(H)] \cup [\bar{\varrho}(B) \cap \varrho(H^0)].$$

If A_1 and B_1 are any other sets from $\mathcal{D}$ such that $E = (A_1 \cap H) \cup \cup (B_1 \cap H^0)$, then $A \Delta A_1 \in \mathcal{T}_1(H)$ and $B \Delta B_1 \in \mathcal{T}_1(H^0)$. Hence

$$\bar{\varrho}(A \Delta A_1) \cap \varrho(H) = \emptyset$$

and $\bar{\varrho}(B \Delta B_1) \cap \varrho(H^0) = \emptyset$, so that the definition of $\varrho(E)$ is correct. It is easy to see that, for every $A \in \mathcal{D}$, we have $\varrho(A) = \bar{\varrho}(A)$.

From now on we assume the continuum hypothesis.

THEOREM 1. *If $\mathcal{A}$ is a σ-algebra and* $\operatorname{card} \mathcal{A}/\mathcal{I} \leqslant \mathfrak{c}$ *(continuum), then there exists a lifting of $\mathcal{A}$ with respect to $\mathcal{I}$.*

Proof. Take from each of the equivalence classes (mod $\mathcal{I}$) one set. Let $\{A_\gamma\}_{\gamma<\omega_1}$ be the family of all such sets ordered by ordinals $\gamma < \omega_1$. We can assume that they are taken in such a way that, for every A_γ and every $B \in \mathcal{D}_\gamma = a(\{A_\beta\}_{\beta<\gamma})$, we have either $A_\gamma \nsim B$ or $A_\gamma \in D_\gamma$.

Let us make the inductive assumption that, for a certain $\gamma < \omega_1$, a family of consistent $(\mathcal{D}_\xi, \mathcal{A})$-liftings ϱ_ξ with respect to $\mathcal{I}$ for all $\xi < \gamma$ is already constructed (i.e., for $\xi_1 < \xi_2$, ϱ_{ξ_1} is the restriction of ϱ_{ξ_2} to $\mathcal{D}_{\xi_1}$).

It is our aim to define a $(\mathcal{D}_\gamma, \mathcal{A})$-lifting ϱ_γ with respect to $\mathcal{I}$ being an extension of all ϱ_ξ for $\xi < \gamma$.

If $\gamma = \gamma_1 + 1$, then the existence of ϱ_γ follows from the Lemma.

If γ is a limit ordinal, then we take as ϱ_γ the common extension of all ϱ_ξ for $\xi < \gamma$; obviously, $\varrho_0(\emptyset) = \emptyset$ and $\varrho_0(X) = X$.

We can see that the mapping $\varrho\colon \mathcal{A} \to \mathcal{A}$, defined by $\varrho(A) = \varrho_\gamma(A_\gamma)$ for $A \sim A_\gamma$, is the required lifting of $\mathcal{A}$ with respect to $\mathcal{I}$.

THEOREM 2. *Let $\mathcal{A}$ be a σ-ring. If* $\operatorname{card} \mathcal{A}/\mathcal{I} \leqslant \mathfrak{c}$ *and $\mathcal{A}/\mathcal{I}$ satisfies the countable chain condition, then there exists a lifting of $\mathcal{A}$ with respect to $\mathcal{I}$.*

Proof. The properties of $\mathcal{A}/\mathcal{I}$ imply that there exists a set $Y \in \mathcal{A}$ such that, for each $A \in \mathcal{A}$, we have $A - Y \sim \emptyset$. In view of Theorem 1, there exists a lifting ϱ of $\mathcal{A} \cap Y$ with respect to $\mathcal{I} \cap Y$.

Putting $\tilde{\varrho}(A) = \varrho(A \cap Y)$ for every $A \in \mathscr{A}$, we get a lifting of $\mathscr{A}$ with respect to $\mathscr{I}$.

COROLLARY 1. *Let $\mathscr{B}_0$ be the σ-algebra of all subsets of a separable metric space X with the Baire property and let $\mathscr{I} \subset \mathscr{B}_0$ be the σ-ideal of all sets of the first category. Then there exists a lifting of $\mathscr{B}_0$ with respect to $\mathscr{I}$.*

COROLLARY 2. *Let $\mathscr{B}$ be the σ-algebra of Borel subsets of a separable metric space X and let μ: $\mathscr{B} \to G$ be a measure with values in a topological abelian group G. If $\mathscr{I} = \{B \in \mathscr{B}$: for every $D \in \mathscr{B} \cap B$ we have $\mu(D) = 0\}$, then there exists a lifting of $\mathscr{B}$ with respect to $\mathscr{I}$ (so called Borel lifting, cf. [1]).*

Added in proof. Professor C. Ryll-Nardzewski has informed us that Theorem 2 remains true without the assumption that $\mathscr{A}/\mathscr{I}$ satisfies the countable chain condition.

REFERENCES

[1] A. Ionescu Tulcea and C. Ionescu Tulcea, *Topics in the theory of lifting*, Berlin, Heidelberg and New York 1969.

[2] J. von Neumann and M. H. Stone, *The determination of representative elements in the residual classes of a Boolean algebra*, Fundamenta Mathematicae 25 (1935), p. 353-378.

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