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INVARIANT EXTENSIONS OF THE HAAR MEASURE

BY

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The first idea of extending the Lebesgue measure in R^n to a larger σ-field in such a way that it remains invariant under translations (or even under all isometric transformations of R^n) seems to belong to Jankowska-Wiatr who in 1928 observed that one can add new sets to the σ-ideal of the sets of Lebesgue measure zero and still preserve the invariance of the extended measure. In 1935 E. Marczewski called attention to this apparently unpublished result and — what has proved to be more important — applied Sierpiński's construction of an almost invariant set A (i.e. a set such that the Lebesgue measure of $A \triangle (A+x)$ is zero for all $x \in R^n$) to obtain a proper extension of the Lebesgue measure to an invariant measure in which the extended σ-field contains new sets of positive finite measure. This pioneering idea has been carried over by Kakutani and Oxtoby [2] who were able to construct a family $\mathscr{A}$ of almost invariant subsets of the circle in such a way that

$$\bigcap_{n}^{\infty} A_n^{\varepsilon_n}$$

has outer measure 1 for an arbitrary sequence $\{A_n\}$ of sets from $\mathscr{A}$ and arbitrary sequence $\{\varepsilon_n\}$, $\varepsilon_n = 0, 1$. Then putting $\overline{m}(A) = 1/2$ for A in $\mathscr{A}$ they obtained an extension of the Lebesgue measure on the circle to an invariant measure $\overline{m}$ such that $L^2(\overline{m})$ has the Hilbert space dimension equal to $2^{\mathfrak{c}}$.

At the same time Kodaira and Kakutani [3] invented another method of extending the Lebesgue measure on the circle to an invariant measure.

The method is to produce a *character* π of the circle T, i.e. a homomorphism $\pi: T \to T$ in such a way that the outer Lebesgue measure of its graph D_π is equal to 1 in $T \times T$. Then the extended σ-field $\overline{\mathscr{B}}$ consists of the sets $A_M = \{x: (x, \pi(x)) \in M\}$, where M is a Lebesgue measurable set in $T \times T$ and the extended measure $\overline{m}$ is $\overline{m}(A_M) = m(M)$. Of course, the discontinuous character π becomes $\overline{\mathscr{B}}$-measurable. It has been noticed later [1] that one can produce 2^c such characters so that they all become measurable and $L^2(\overline{m})$ is of Hilbert space dimension 2^c.

Of course, one can combine all three methods: enlarging the σ-ideal of sets of measure zero, adding some almost invariant sets and making new characters measurable, and these seem to be the most general invariant extensions of the Lebesgue measure on the circle T known.

The aim of this note is to characterize these extensions in terms of the action of the group T on $L^2(\overline{m})$. Since the result easily generalizes to any compact group with the Haar measure in place of the Lebesgue measure, we present it in this generality.

Suppose that G is a compact group, m its normalized Haar measure defined on the σ-field $\mathscr{B}$ of Haar measurable sets and let $(G, \overline{\mathscr{B}}, \overline{m})$ be an extension of (G, B, m) to an invariant measure. Let further $x \to L_x$ be the left regular representation of G on $L^2(\overline{m})$, i.e., for f in $L^2(\overline{m})$, $L_xf(y) = f(x^{-1}y)$, $x, y \in G$.

THEOREM. *The following statements are equivalent:*

(i) *For every f in $L^2(\overline{m})$ the vector-valued function*

$$G \ni x \mapsto L_xf \in L^2(\overline{m})$$

is $\overline{\mathscr{B}}$-measurable and the set $\{L_xf\colon x \in G\}$ is relatively compact in $L^2(\overline{m})$.

(ii) *The σ-field $\overline{\mathscr{B}}$ is generated (as a σ-field) by a family of left almost invariant sets with respect to $\overline{m}$ and by co-images of open sets by a family of $\overline{\mathscr{B}}$-measurable finite-dimensional unitary representations of G.*

Proof. Let π be a finite-dimensional unitary representation of G which is $\overline{\mathscr{B}}$-measurable. By a *representing function* connected with π we mean a function which is of the form

$$a(x) = \big(\pi(x)f, g\big),$$

where f and g are vectors from the space on which π acts. By definition, the function a is $\overline{\mathscr{B}}$-measurable. A function $f \in L^2(\overline{m})$ is called *almost invariant* if, for every x in G, $L_xf(y) = f(y)$ for almost all y, i.e. $L_xf = f$.

To prove (i) $\Rightarrow$ (ii) it is sufficient to show that there is a Hilbert space basis of $L^2(\overline{m})$ which consists of functions which are linear combinations of products of representing functions and almost invariant functions.

We start with a simple lemma which easily follows from what are classical facts about almost periodic functions on groups (cf. [5]) and we supply the proof here for completeness sake.

LEMMA. *Suppose that G is a discrete group and π a unitary representation of G on a Hilbert space H such that for every f in H the orbit $O_f = \{\pi(x)f\colon x \in G\}$ is relatively compact. Then there exist a compact group $\overline{G}$, a continuous unitary representation $\overline{\pi}$ of $\overline{G}$ and a homomorphism $a\colon G \to \overline{G}$ such that $\pi = \overline{\pi} \circ a$.*

Proof. By passing to a factor group, if necessary, we assume that π is faithful, i.e. $\pi(x) = \mathrm{id}$ implies $x = e$. We equip G with the strong operator topology by defining the neighbourhoods of e by

$$U_{f_1,\ldots,f_n,\varepsilon} = \{x\colon \|\pi(x)f_j - f_j\| < \varepsilon,\ j = 1, \ldots, n\}.$$

Since $\|\pi(x)\| = 1$, we have

$$\begin{aligned}\|\pi(xy)f-f\| &\leqslant \|\pi(x)(\pi(y)f-f)\| + \|\pi(x)f-f\| \\ &\leqslant \|\pi(y)f-f\| + \|\pi(x)f-f\|,\end{aligned}$$

whence

$$U_{f_1,\ldots,f_n,\varepsilon/2}\cdot U_{f_1,\ldots,f_n,\varepsilon/2} \subset U_{f_1,\ldots,f_n,\varepsilon},$$

which shows that this is a group topology.

On the other hand, since O_f, $f \in H$, are relatively compact, G is totally bounded. Indeed, for every $U_{f_1,\ldots,f_n,\varepsilon}$, if $x_1, \ldots, x_k$ are such that for each x in G and $j = 1, \ldots, n$ we have

$$\|\pi(x)f_j - \pi(x_i)f_j\| < \varepsilon \quad \text{for some } i = 1, \ldots, k,$$

then

$$G \subset \bigcup_{j=1}^{k} x_j U_{f_1,\ldots,f_n,\varepsilon}.$$

The completion $\bar{G}$ of G in this topology is a compact group and, since by definition $G \ni x \to \pi(x)f \in H$ is continuous, it extends uniquely to a continuous function $\bar{G} \ni x \mapsto \bar{\pi}(x)f \in H$, which defines the desired representation $\bar{\pi}$.

COROLLARY. *If π is a unitary representation of a group G on a Hilbert space H such that $O_f = \{\pi(x)f\colon x \in G\}$ are relatively compact, then H is an orthogonal direct sum of finite-dimensional G-invariant subspaces.*

Back to the proof of (i) $\Rightarrow$ (ii), we apply this corollary to the left regular representation of G on $L^2(\overline{m})$.

Let $e_1, \ldots, e_n$ be an orthonormal basis of a finite-dimensional invariant subspace of $L^2(\overline{m})$. We have

$$L_x e_i = \sum_j a_{ij}(x) e_j.$$

By assumption, the representing functions a_{ij} are $\overline{m}$-measurable, whence $a_{ij} \in L^\infty(\overline{m})$. Since L is a unitary representation, the matrix $(a_{ij}(x))$ is unitary for every x in G and

$$a_{ij}(xy) = \sum_k a_{ik}(y) a_{kj}(x).$$

Consider the functions

$$\varphi_j(x) = \sum_i e_i(x) a_{ji}(x), \quad j = 1, \ldots, n.$$

We have

$$
\begin{aligned}
(L_y\varphi_j)(x) &= \sum_i e_i(y^{-1}x)\,a_{ji}(y^{-1}x)\\
&= \sum_{i,k,l} a_{ik}(y)\,e_k(x)\,a_{jl}(x)\,a_{li}(y^{-1})\\
&= \sum_k e_k(x)\,a_{jk}(x) = \varphi_j(x).
\end{aligned}
$$

This shows that functions $\varphi_1, \ldots, \varphi_n$ are almost invariant and, since

$$e_i(x) = \sum_j \varphi_j(x)\,a_{ji}(x),$$

implication (i) $\Rightarrow$ (ii) follows.

Now assume that (ii) holds. To prove (i) we note that it is sufficient to prove it for a dense subset of functions in $L^2(\overline{m})$. Therefore we are going to prove that the set F of functions f in $L^2(\overline{m})$ such that $x \to L_x f$ is $\overline{m}$-measurable and $O_f = \{L_x f\colon x \in G\}$, as relatively compact, is dense. Clearly enough, F contains the set Φ consisting of all representative functions connected with finite-dimensional unitary $\overline{m}$-measurable representations of G and almost invariant functions in $L^2(\overline{m})$, as well the $*$-algebra of functions generated by Φ.

Thus to prove the implication (ii) $\Rightarrow$ (i), it suffices to prove the following

PROPOSITION. *Let $(X, \overline{\mathscr{B}}, \overline{m})$ be a measure space with $\overline{m}(X) < \infty$. Let Φ be a family of functions in $L^\infty(\overline{m})$. If the smallest σ-subfield of $\overline{\mathscr{B}}$ with respect to which all functions in Φ are measurable is equal to $\overline{\mathscr{B}}$, then the σ-algebra generated by Φ is dense in $L^2(\overline{m})$.*

Proof. Since the smallest norm-closed $*$-algebra of $L^\infty(\overline{m})$ containing Φ has the same smallest σ-subfield with respect to which all functions are measurable as Φ does, we may assume that Φ already is a norm-closed $*$-subalgebra of $L^\infty(\overline{m})$. Let $\mathscr{K}$ be the σ-field of Borel subsets of the complex plane C. By assumption $\overline{\mathscr{B}}$ is the smallest σ-field containing all sets of the form $f^{-1}(M)$, where $M = \overline{M} \subset C$, $f \in \Phi$. Thus to prove the proposition it is sufficient to show that the characteristic function $1_{f^{-1}(M)}$ of the set $f^{-1}(M)$ is a limit of a decreasing sequence of functions from Φ. This, however, is easy: since $1_{f^{-1}(M)} = 1_M \circ f$ and 1_M is a limit of a decreasing sequence of continuous functions φ_n and $\varphi \circ f \in \Phi$ for every continuous function φ, we have

$$1_{f^{-1}(M)} = \lim_n \varphi_n \circ f(x)$$

and the proposition follows.

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