COLLOQUIUM MATHEMATICUM

VOL. XIX 1968 FASC. 1

ON THE UNITARY REPRESENTATIONS OF ABELIAN LOCALLY COMPACT GROUPS

BY

R. JAJTE (ŁÓDŹ)

Let T denote an abelian locally compact group and let H be a real separable Hilbert space. Let $\mathfrak{M}$ be a set of all probability distributions in H (i.e., the set of regular normed measures defined on a σ-field $\mathfrak{B}$ of Borelian subsets of H). With the metric of Lévy-Prochorov ([4], p. 188) $\mathfrak{M}$ is a complete metric space. Convergence with respect to that metric is equivalent to the weak convergence of distributions (1). The distance between μ and ν in $\mathfrak{M}$ we denote by $|\mu, \nu|$. For a linear bounded operator A in H and for $\mu \in \mathfrak{M}$ we put

(1) $$(A\mu)(Z) = \mu(A^{-1}Z) \quad \text{for any } Z \in \mathfrak{B}.$$

The distribution

(2) $$(\mu * \nu)(Z) = \int \mu(Z-h)\nu(dh) \quad \text{for any } z \in \mathfrak{B}$$

is called the *convolution* of distributions μ and ν (2). The convolution of distributions $\mu_1, \mu_2, \ldots, \mu_n$ will be denoted by $\prod_{k=1}^{n}{}^{*} \mu_k$. By δ_{x_0} we denote the measure condensed at point x_0, i.e. $\delta_{x_0}(Z) = 1$ if $x_0 \in Z$ and $\delta_{x_0}(Z) = 0$ if $x_0 \notin Z$.

We shall prove the following

THEOREM. *Let μ be a probability distribution in H satisfying the condition*

(3) $$\int \|h\|^2 \mu(dh) < \infty$$

and let $(U_t, t \in T)$ be a continuous unitary representation of abelian locally compact group T in H. Then there exists an element $h_0 \in H$ such that for

(1) A sequence of measures $\{\mu_n\}$ is said to be *weakly convergent* to μ ($\mu_n \to \mu$) if for any continuous function bounded in H we have

$$\int f(h)\mu_n(dk) \to \int f(h)\mu(dh) \quad (n \to \infty).$$

(2) $\int \ldots$ means integral over the whole space H.

any $\varepsilon > 0$ *there exists a system* $t_1, t_2, \ldots, t_N$ *of elements of* T *such that*

$$\sup_{t \in T} \left| \prod_{k=1}^{N}{}^{*} \frac{1}{N} U_{tt_k}\mu, \delta_{h_0} \right| < \varepsilon .$$

First we prove the following two lemmas.

LEMMA 1. *If* $(U_t, t \in T)$ *is a continuous unitary representation of an abelian locally compact group* T *in Hilbert space* H, *then for every* $h \in H$ *there exists an element* $m(h) \in H$ *such that for any* $\varepsilon > 0$ *there exists a system* $t_1, t_2, \ldots, t_N$ *of elements of* T *such that*

$$\sup_{t \in T} \left\| \frac{1}{N} \sum_{k=1}^{N} U_{tt_k} h - m(h) \right\| < \varepsilon . \tag{4}$$

This lemma is proved in [1] in particular case $H = L_2(\Omega, \mathfrak{A}, p)$, where $(\Omega, \mathfrak{A}, p)$ is a probability space. The proof may be easily transfered to the general case of abstract Hilbert space.

LEMMA 2. *If the distribution* μ *in* H *satisfies condition* (3) *and the mathematical expectation of* μ (3) *is equal to zero, then for every* $\varepsilon > 0$ *there exists a natural number* m *such that*

$$\left| \prod_{i=1}^{n}{}^{*} \frac{1}{n} U_i \mu, \delta_0 \right| < \varepsilon \tag{5}$$

for each $n \geq m$ *and for every system of unitary operators* $U_1, U_2, \ldots, U_n$ *in* H.

Proof. Let $\hat{\mu}$ be the characteristic functional of the distribution μ, i.e. (see [2], [3], [4])

$$\hat{\mu}(h) = \int e^{i(g,h)} \mu(dg). \tag{6}$$

From (3) and $M_\mu = \theta$ follows

$$\hat{\mu}(h) = 1 - \tfrac{1}{2}(Dh, h) + o(\|h\|^2), \tag{7}$$

where D is the dispersion operator of μ defined by the formula (see [4])

$$(Dg, h) = \int (g, u)(h, u) \mu(du). \tag{8}$$

The characteristic functional of the probability measure

$$\nu_n = \prod_{i=1}^{n}{}^{*} \frac{1}{n} U_i \mu \tag{9}$$

(3) An element $M_\mu \in H$ such that $(M_\mu, h) = \int (g, h) \mu(dg)$ is called the *mathematical expectation* of a probability measure μ (see [3] and [4]).

is of the form

$$\hat{\nu}_n(h) = \prod_{i=1}^{n} \hat{\mu}\left(\frac{1}{n} U_i^{-1} h\right). \tag{10}$$

Thus from (7) we have

$$\hat{\nu}_n(h) = \prod_{i=1}^{n}\left[1 - \frac{1}{2n^2}(DU_i^{-1}h,\, U_i^{-1}h) + \frac{1}{n^2}\|h\|^2 \alpha\left(\frac{1}{n} U_i^{-1}h\right)\right], \tag{11}$$

where $\alpha(h) \to 0$ as $h \to \theta$. It easily follows from formula (11) that

$$\lim_{n\to\infty} \hat{\nu}_n(h) = 1 = \hat{\delta}_\theta(h) \tag{12}$$

for every $h \in H$. In order to prove that $\nu_n \to \delta_\theta$, it suffices to show that the sequence

$$D_n = \frac{1}{n^2} \sum_{k=1}^{n} U_k D U_k^{-1} \quad (n = 1, 2, \ldots) \tag{13}$$

of dispersion operators of distributions ν_n satisfies the following conditions (see [4], § 4):

$$\sup_n \operatorname{Tr} D_n < \infty, \tag{14a}$$

where Tr D denotes the trace of D,

$$\lim_{m\to\infty} \sup_n \sum_{i=m}^{\infty} (D_n e_i, e_i) = 0, \tag{14b}$$

where $\{e_i\}$ is a basis in H.

This is easy to verify and may be omitted.

Since the weak convergence of distributions is equivalent to the convergence with respect to the metric in $\mathfrak{M}$, our lemma is proved.

Proof of the theorem. Let M denote the mathematical expectation of distribution μ. We have $\mu = \mu * \delta_{-M} * \delta_M$ and the mathematical expectation of the distribution

$$\nu = \mu * \delta_{-M} \tag{15}$$

is equal to zero. Moreover, we have

$$\prod_{k=1}^{N}{}^{*} \frac{1}{N} U_{tt_k}\mu = \left(\prod_{k=1}^{N}{}^{*} \frac{1}{N} U_{tt_k}\nu\right) * \delta_{\frac{1}{N}\sum_{k=1}^{N} U_{tt_k}M}\,. \tag{16}$$

The distribution ν satisfies the conditions of lemma 2. In virtue of lemma 1, for a given $\eta > 0$ there exists $h_0 \in H$ and a system $t_1, t_2, \ldots, t_N$ of elements of the group T such that

$$\sup_{t\in T} |\delta_{\frac{1}{N}\sum_{k=1}^{N} U_{tt_k}M}\;\delta_{h_0}| < \eta. \tag{17}$$

We can choose N in (17) such that

$$\sup_{t\in T}\left|\prod_{k=1}^{N}{}^{*}\frac{1}{N}U_{tt_k}\nu,\ \delta_0\right| < \eta \tag{18}$$

(in virtue of lemma 2). The convolution of measures is continuous with respect to the metric in $\mathfrak{M}$, thus for any $\varepsilon > 0$ we can choose an $\eta > 0$ such that (17) and (18) implies

$$\sup_{t\in T}\left|\prod_{k=1}^{N}{}^{*}\frac{1}{N}U_{tt_k}\mu,\ \delta_{h_0}\right| < \varepsilon.$$

This completes the proof of the theorem.

REFERENCES

[1] R. Jajte, *On stochastic processes over an abelian locally compact group,* Colloquium Mathematicum 17 (1967), p. 351-355.

[2] A. N. Kolmogorov, *La transformation de Laplace dans les espaces linéaires,* Comptes Rendus Acad. Sci. Paris 200 (1935), p. 1717.

[3] E. Mourier, *Eléments aléatoires dans un space de Banach,* Annales Inst. Henri Poincaré 13 (3) (1953), p. 161-244.

[4] Ю. В. Прохоров, *Сходимость случайных процессов и предельные теоремы теории вероятностей,* Теория вероятностей и её применения 1 (1956), p. 177-238.

Reçu par la Rédaction le 1. 2. 1967