COLLOQUIUM MATHEMATICUM

VOL. XIX 1968 FASC. 1

ON FINITE-DIFFERENCE APPROXIMATIONS TO STEADY-STATE SOLUTIONS OF THE NAVIER-STOKES EQUATIONS

BY

A. KRZYWICKI (WROCŁAW)

In papers [2] and [3] some convergent finite-difference approximations to solutions of the non-stationary Navier-Stokes equations have been given. We shall show here that a similar construction also for steady-state solutions of the same equations is possible, more precisely, for solutions of the following boundary value problem

$$(1)\qquad -\nu\Delta\mathcal{U}+\sum_i \mathcal{U}^i\frac{\partial\mathcal{U}}{\partial x^i} = -\nabla\mathcal{P}+\mathcal{F},\quad \nabla\cdot\mathcal{U}=0,\quad \mathcal{U}|_S=\Psi,$$

considered in a domain Ω of the Euclidean 2- or 3-dimensional space. Here ν is a positive constant, S denotes the boundary of Ω, $\mathcal{U}$ (vector) and $\mathcal{P}$ (scalar) are unknown, $\mathcal{F}$ and Ψ are given vectors defined on Ω and S resp. (we shall constantly use the term *vector* when saying of vector-valued functions.) $\mathcal{U}^i$ denotes the i-th component of vector $\mathcal{U}$ and finally x^i are coordinates of the point x.

The principal purpose of this paper is to establish the applicability of some of arguments of papers [2] and [3] to the case under consideration. Therefore we do not insist on considering a possibly most general situation. All results of the present paper are equally valid in 2- and 3-dimensional cases. However, for the sake of simplicity, we discuss 2-dimensional case only, pointing out necessary modifications, if needed, when passing to 3-dimensional space. It will be clear to the reader that some of the results presented here are finite-difference modifications of known facts: (7) corresponds to the known decomposition of $L_2(\Omega)$-space, the proof of Theorem 1 is a modification of Fujita's procedure [1], and Theorem 5 is a finite-difference analogue of a result of Ladyženskaja [4]. To avoid some unnecessary repetitions we shall often refer to paper [2] which precedes immediately this paper in the present fascicule. We shall also use most of notations of [2] (see especially section 2 of [2]).

In particular, the functions considered on sets of grid points will be denoted with small letters. The inner product, scalar multiplication, orthogonality, if referred to functions considered on a grid, will be understood in the sense of definition (3) of [2]. When referred to "normal" functions, they will be understood in the usual $L_2(\Omega)$-space sense. Different finite-difference approximations introduced in [2] will be also used here and will be denoted with the same letters as in [2].

1. Consider in E^2 an orthogonal grid G_h of points with coordinates $x^i = \pm nh$, h denoting a positive constant, $n = 0, 1, 2, \ldots$ Given a domain $\Omega \subset E^2$, let Ω_h be the union of all *elementar squares*, i.e. closed squares with corners on G_h and sides equal to h, lying in Ω. The set $\Omega_h \cap G_h$ will be denoted by ω, the set of all grid points lying on the boundary of Ω_h will be called *boundary* of ω and denoted by s. The sets of grid points lying on individual components of the boundary of Ω_h will define components s_i of boundary of s. Finally, the set $\omega^{int} = \omega \setminus s$ will be called *interior* of ω.

A vector $v = (v^1, v^2)$, due to our convention of notation defined on G_h, will be called *solenoidal*, if it satisfies the equation $\operatorname{div} v \equiv v^1_1 + v^2_2 = 0$ (remember that $f_i = h^{-1}(\overset{+i}{f} - f)$, see [2]). Solenoidal vectors of a special kind will be of use in the sequel. They will be denoted by $r = r(x - x')$ and their components will be defined in the following way:

$$r^1 = \delta_2 = h^{-1}(\overset{+2}{\delta} - \delta), \qquad r^2 = -\delta_1 = -h^{-1}(\overset{+1}{\delta} - \delta),$$

where $\delta = \delta(x - x')$, $x, x' \in G_h$, denotes the function which is zero everywhere except for the point x' where $\delta = 1$. $r(x - x')$ is different from zero on the set $\operatorname{supp} r$ consisting of three points $\{x', x' - he^1, x' - he^2\}$ upon which it assumes the values $(-1, 1)$, $(0, -1)$ and $(1, 0)$, resp. (e^i denotes the unit vector of the x^i-axis). Each r is solenoidal and r's corresponding to different points x' are linearly independent.

A line Γ consisting of sides of elementar squares will be called simply *line* (strictly speaking only the grid points lying on Γ will be considered). Γ provided with an orientation will be called *oriented line*. It consists of oriented segments γ of the grid of length h, each γ defined by a segment $[x_\gamma - he^{i_\gamma}, x_\gamma]$ and the number ε_γ, $\varepsilon_\gamma = 1$ if the orientations of γ and of the x^i-axis are the same, $\varepsilon_\gamma = -1$ if opposite. For a given vector v and a given oriented line $\Gamma = \bigcup_\gamma \gamma$ we can form the expression

$$I(v, \Gamma) = h \sum_\gamma \varepsilon_\gamma v^{i_\gamma}(x_\gamma) \tag{2}$$

which imitates the curvilinear integral. If Γ is an oriented line joining the points $x^{(0)}$ and x, the orientation being from $x^{(0)}$ to x, then $I(v, \Gamma)$, considered as a function $\Phi(x)$ of x, satisfies the equation $\Phi_{\bar{i}} = v^i$, or simply grad $\Phi = v$, where grad $\Phi = (\Phi_{\bar{1}}, \Phi_{\bar{2}})$.

Finally, a line Γ will be said to *lie in the set* ω if all its grid points lie in ω.

2. Assume that the domain $\Omega \subset E^2$, where the problem (1) is considered, is bounded and the number of components of its boundary is finite. Assume further that the set ω^{int} which was defined before is for any h connected which means that each pair of its points may be joined by means of a line lying in it. Assume, moreover, that ω^{int} is of the form

$$\omega^{int} = \bigcup_{x'} \operatorname{supp} r(x-x').$$

If not, we could change Ω_h by casting away some of its elementar squares so that ω^{int} corresponding to the new Ω_h' would be of the desired form. We should then assume that mes $(\Omega \setminus \Omega_h') \to 0$ when $h \to 0$.

Denote by $\mathring{j}(\omega)$ the class of all solenoidal vectors defined on ω and vanishing on s (solenoidal in ω means: solenoidal in all points of ω where div may be applied).

LEMMA. *If a vector v defined on ω is orthogonal to all solenoidal vectors vanishing on s, i.e. if*

$$(v, w) = h^2 \sum vw = 0 \tag{3}$$

for any $w \in \mathring{j}(\omega)$, then $v =$ grad Φ in ω^{int}, where Φ is a function defined on the set

$$\omega^* = \bigcup_{x' \in \omega^{int}} \operatorname{supp} r(x-x')$$

up to an additive constant.

Proof. Put $w = r \in \mathring{j}(\omega)$ into (3). The result, when explicitely written down, has the form of the equation

$$v^1_{\bar{2}} = v^2_{\bar{1}} \tag{4}$$

valid in any point x' such that $\operatorname{supp} r(x-x') \subset \omega^{int}$. It follows from (4) that $I(v, \Gamma)$, defined by (2), vanishes if Γ is (arbitrarily oriented) boundary of an elementary square. Fix a point $x^{(0)} \in \omega^{int}$ and join it with any other point $x \in \omega^{int}$ by means of an oriented line Γ lying in ω^{int}, its orientation being from $x^{(0)}$ to x. $I(v, \Gamma)$ is, due to the preceding remark, a function $\Phi(x)$ of x only in any simply connected part of ω^{int} containing $x^{(0)}$ and grad $\Phi = v$ due to the definition of $I(v, \Gamma)$. To show that $\Phi(x)$ is single valued also in the case when ω^{int} is multiply connected we may proceed as in the similar situation in "continuous" case. Let us cut ω^{int} by means of lines so as to get a simply connected set ω_c and a single valued function Φ_c on it. Consider one of the cutting lines Γ_c and assume that it contains segment $[x''-h\mathbf{e}^2, x''+2h\mathbf{e}^2]$. Assume for a moment that there exists a solenoidal vector w which vanishes outside of a given connected set lying in ω^{int} and having the form of a ring. Let, moreover, w be zero along Γ_c except for the point x'' where w^1 is different from zero, say $w^1(x'') = 1$. For any $v \in \overset{\circ}{j}(\omega)$ and any ψ we have

$$(5) \qquad (\operatorname{grad}\psi, v) = -(\psi, \operatorname{div} v) = 0,$$

and the values of ψ involved in (5) are those taken in points of ω^* only. If we apply the above formula to Φ_c constructed before and take into consideration the cut of T_c then, putting $v = w$, we get

$$0 = (\operatorname{grad}\Phi_c, w) = -(\Phi_c, \operatorname{div} w) + w^1(x'')(\Phi^- - \Phi^+),$$

where Φ^- and Φ^+ denote the values of Φ_c at the point x'' when we approach it from the left or right side resp. Hence we get, due to the properties of w, $\Phi^- = \Phi^+$ and so Φ_c has no jump when passing through Γ_c. This shows that Φ is a single-valued function in all ω^{int}. It immediately follows from the definition that Φ may be extended to ω^*.

We shall now establish the existence of the vector w with needed properties by making use of the known similar construction in "continuous" case: there exists, for any given simple connected domain $\mathcal{T}_c$ having the form of a ring cutted by l, a regular divergence free vector W vanishing outside of $\mathcal{T}_c$ and assuming along the cut l (which presents two different parts of the boundary of $\mathcal{T}_c$) the same arbitrarily given values (see [4], where the 3-dimensional case is discussed). It suffices now to suppose that the total flux of W through l is equal to 1, and that $\operatorname{supp} W$ has a special form near the point x'': it lies in a narrow strip running parallelly to the x^1-axis and crossing $\mathcal{T}_c$ between points x'' and $x''+h\mathbf{e}^2$. Moreover, $\operatorname{supp} W$ should be not too close to s. The desired w will be given by a finite difference approximation of W provided by the ormulas

$$(6) \qquad w^1(x) = h^{-1}\int_{x^2}^{x^2+h} W^1(x^1, \xi)\,d\xi, \qquad w^2(x) = h^{-1}\int_{x^1}^{x^1+h} W^2(\xi, x^2)\,d\xi.$$

To end the proof of Lemma it remains only to show that the function Φ is defined up to an additive constant. This is obvious: if $v = \operatorname{grad}\Phi = \operatorname{grad}\Phi'$, then it follows that $\operatorname{grad}(\Phi-\Phi') = 0$ on ω^{int} and hence $\Phi = \Phi' + \text{const}$ due to the assumed connectivity of ω^{int}.

Lemma in connection with (5) leads to the following result: any vector u defined on ω may be expressed on the set ω^{int} in the form

$$(7) \qquad u = v + \operatorname{grad}\Phi,$$

where $v \in \overset{\circ}{j}(\omega)$ and Φ is a function defined on ω^*. The decomposition given by (7) is unique, the function Φ being defined up to an additive constant.

3. The following two inequalities

$$\|u\| \leqslant c_0 \|u_x\|, \tag{8}$$

$$h^2 \sum |u|^4 \leqslant c_1^2 \|u\|^2 \|u_x\|^2, \tag{9}$$

are valid for any vector u defined on ω and vanishing on s. The constants c_0 and c_1 are independent of u and h and in the case of c_1 — even of ω. $|u|$ denotes Euclidean length of vector u and the remaining notation is the same as in [2]. The both inequalities present finite-difference analogues of known inequalities (see [4], p. 18-19) and may be easily derived by adapting the proofs of the latters to our case.

4. We define for any triple u, v, w of vectors of the class $\overset{\circ}{j}(\omega)$ the following trilinear form

$$A(u, v, w) = \tfrac{1}{2} h^2 \sum \sum_{i,j} u^i v^j (w^i_{\,j} + w^i_{\,\bar{j}}), \tag{10}$$

the first summation in (10) being, as usually, over all those points of ω where the summand is defined. Applying identities (11) and (13) of [2] and the identity $(uv)_{\bar{i}} = u_{\bar{i}} \overset{-i}{v} + uv_{\bar{i}}$, we easily verify that A satisfies the following identity:

$$A(u, v, w) = -A(w, v, u). \tag{11}$$

In particular,

$$A(u, v, u) = 0 \tag{12}$$

for any $u, v \in \overset{\circ}{j}(\omega)$. Applying twice Cauchy's inequality to the form A, we get

$$|A(u, v, w)|^4 \leqslant \left(h^2 \sum |u|^4\right) \left(h^2 \sum |v|^4\right) \|w_x\|^4,$$

and hence, due to (9),

$$|A(u, v, w)| \leqslant \tfrac{1}{2} c_1 (\|u\| \|u_x\| + \|v\| \|v_x\|) \|w_x\|. \tag{13}$$

In particular, relations (13) and (8) imply the inequalities

$$|A(u, v, w)| \leqslant \tfrac{1}{2} c_0 c_1 (\|u_x\|^2 + \|v_x\|^2) \|w_x\|, \tag{14}$$

$$|A(u, u, w)| \leqslant c_0 c_1 \|u_x\|^2 \|w_x\|, \tag{15}$$

valid for any $u, v, w \in \overset{\circ}{j}(\omega)$.

Remark. In 3-dimensional space inequality (8) remains unchanged (with a different constant only) and (9) is to be replaced by

$$h^3 \sum |u|^4 \leqslant c_2^2 \|u\| \, \|u_x\|^3 .$$

However, the form of estimates (14) and (15) remains unchanged; therefore all the consequences of these estimates, in particular, Theorem 5 remain valid also in 3-dimensional case.

5. We assume that the vector Ψ appearing in the boundary condition imposed on the solution $\mathscr{U}$ of problem (1) is the value along S of a vector $\mathscr{B}$ defined and divergence free in a domain containing Ω in its interior. Let b denote the approximation of $\mathscr{B}$ given by formula (4) of [2] (or by formulas (6) of the present paper if $\mathscr{B}$ is regular). The vector $\mathscr{F}$ appearing in (1) is assumed to belong to $L_2(\Omega)$. Its approximation constructed in the same way as it was done in [2] (end of section 2) will be denoted by f.

We shall now discuss a straightforward difference substitute of the problem (1), namely the following system of non-linear finite-difference equations

$$\sum_i \{-\nu u_{\bar{i}i} + \tfrac{1}{2} u^i (u_i + u_{\bar{i}})\} = -\operatorname{grad} p + f, \tag{16}$$

$$\operatorname{div} u = 0, \tag{17}$$

with unknowns u, p, and u subject to the boundary condition

$$u|_S = b|_S. \tag{18}$$

Equations (16) are taken in all points of ω^{int} whereas equations (17) in all points of the set ω^* defined before. Note that the set ω^*, though now differently defined, is identical with ω^* introduced in [2], p. 147. We attach to our system an additional equation

$$\sum_{\omega^*} p = 0 \tag{19}$$

(and only this one due to the assumption that ω^{int} is connected, comp. a similar situation in [2]).

6. We shall prove the following

THEOREM 1. *If h is fixed and b appearing in* (18) *satisfies for any $w \in \overset{\circ}{j}(\omega)$ the condition*

$$|A(b, w, w)| \leqslant a \|w_x\|, \tag{20}$$

where a is a positive constant less than ν, then the system of equations (16)-(19) *has for any b, f at least one solution u, p with u defined on ω and p on ω^*.*

Proof. Let the vectors $\psi^{(i)}$, $i=1,2,\dots,N$, form an orthonormal basis in $\overset{\circ}{j}(\omega)$, $(\psi^{(i)},\psi^{(j)})=\delta_{ij}$, δ_{ij} denoting Kronecker's delta. Assume u be of the form

$$u=\sum_{i=1}^{N}\xi_i\psi^{(i)}+b, \tag{21}$$

where ξ_i are real coefficients to be defined. Taking the inner product (in the sense of formula (3) of [2]) of both sides of (16) with $\psi^{(i)}$, we get the equations

$$(u_x,\psi_x^{(i)})+A(\psi^{(i)},u,u)=(f,\psi^{(i)}),\qquad i=1,2,\dots,N, \tag{22}$$

which after replacing u by the right-hand member of (21) and looking at $\xi_1,\xi_2,\dots,\xi_N$ as unknowns, become identical with those discussed by Fujita [1] and which have under condition (20) (which is to be applied exactly as in [1], p. 70) at least one solution $\xi_1,\xi_2,\dots,\xi_N$. The last assertion is equivalent to the existence of u in ω satisfying equations (22) and this, on the other hand, expresses the fact that the vector

$$g=\sum_i\{-\nu u_{\bar{i}\bar{i}}+\tfrac{1}{2}u^i(u_i+u_{\bar{i}})\}-f \tag{23}$$

is orthogonal to $\overset{\circ}{j}(\omega)$. Due to Lemma 2, g is in ω^{int} of the form $\operatorname{grad}p$ with p, due to (19), uniquely defined on ω^*. Thus there have been found u and p satisfying equations (16)-(19).

Condition (20) is satisfied in the following two cases:

1° if either (i) $\max\limits_{\omega}|b|$ or (ii) $\max\limits_{\omega}|b_x|$ is sufficiently small. This results when estimating $A(b,w,w)$ by using (i) or $-A(w,w,b)$ ($=-A(b,w,w)$ due to (11)) by using (ii) and then applying Cauchy's inequality and the estimate (8);

2° if the vector b is a linear combination of vectors r. This corresponds to the case when

$$\mathscr{B}=\left(\frac{\partial\Phi}{\partial x^2},-\frac{\partial\Phi}{\partial x^1}\right)$$

and in 3-dimensional space to the case when $\mathscr{B}=\operatorname{rot}\mathscr{D}$.

To show that in the case 2° condition (20) is satisfied, let us note at first that the vectors of $\overset{\circ}{j}(\omega)$ and those r's whose supports do not lie in ω^{int} (the class of these vectors will be denoted by R_0) are linearly independent. Therefore assumption (21) may be replaced by

$$u=\sum\xi_i\psi^{(i)}+\sum_{r\in R_0}\beta_r r,$$

where r's in the last sum are those only whose supports intersect the boundary s of ω. Condition (21) takes now the form

$$\Big|\sum_r \beta_r A(r, w, w)\Big| \leqslant \alpha \|w_x\|. \tag{24}$$

Each term $A(r, w, w)$ of the above sum is equal to the sum extended over (three) points x'' lying in supp r of terms of the form

$$\tfrac{1}{2}h^2 \sum_{i,j} w^j (w^i_{\;j} + w^i_{\;\bar{j}}) \tag{25}$$

provided $x'' \notin s$, If $x'' \in s$, (25) vanishes. Now to each $x'' \notin s$ there exists a point lying on s at the distance h where, due to our assumption, $w = 0$, and therefore (25) may be rewritten as

$$\pm\tfrac{1}{2}h^3 \sum_{i,j} w^j_{\;k} (w^i_{\;j} + w^i_{\;\bar{j}}).$$

Hence the left-hand side member of (24) may be estimated by $Ch\|w_x\|^2$, where C depends on $\max\limits_r |\beta_r|$, and (24) will be satisfied for sufficiently small h with α independent on h if $\max\limits_r |\beta_r|$ will be uniformly bounded with respect to h. The last assumption will be certainly satisfied if

$$\mathscr{B} = \left(\frac{\partial \Phi}{\partial x^2}, -\frac{\partial \Phi}{\partial x^1}\right) \quad \text{with} \quad \Phi \in C^1.$$

To show this, note that the approximation b of $\mathscr{B}$ may be constructed in the following way: we approximate Φ by

$$\varphi(x) = \sum_{x'} \Phi(x')\,\delta(x - x'),$$

the last sum being extended over grid points, and then put $b = (\varphi_2, -\varphi_1)$. The coefficients β_r corresponding to this b will be equal, as is easily seen, to $\pm\varphi_i = \pm h^{-1}(\overset{+i}{\varphi} - \varphi)$ and thus

$$\max_r |\beta_r| \leqslant \left|\frac{\partial \Phi}{\partial x^1}\right| + \left|\frac{\partial \Phi}{\partial x^2}\right| + 1$$

for sufficiently small h.

7. By applying a standard device we obtain an estimate for $\|u_x\|$. To this purpose we multiply equations (22) by corresponding ξ_i and then sum up the results over i. In this way we get

$$\nu(u_x, v_x) + A(v, u, u) = (f, v),$$

where $v = u - b \in \overset{\circ}{j}(\omega)$. This may be rewritten, if we replace u by $v+b$ and then use (11) and (12), in the form

$$\nu \|v_x\|^2 - A(b, v, v) = (f, v) - \nu(v_x, b_x) + A(b, b, v).$$

From the last identity, applying (20) and Cauchy's inequality, we get the estimate of the form

$$\|u_x\| \leqslant C(\nu - a)^{-1} \tag{26}$$

with a constant C depending only on the norms $\|f\|$, $\|b_x\|$, the dependence being so that C tends to zero when these norms do.

THEOREM 2. *The solution* u, p *provided by Theorem* 1 *is unique if only*

$$\|u_x\| < \nu(c_0 c_1)^{-1}, \tag{27}$$

where c_0 *and* c_1 *are the constants appearing in inequalities* (8) *and* (9).

Due to the remark preceding Theorem 2 formula (27) will hold if, for example, $\|f\|$ and $\|b_x\|$ are sufficiently small.

Proof. Let both pairs u, p and u', p' satisfy equations (16)-(19). Then $w = u - u'$, $q = p - p'$ satisfy the equation

$$\sum_i \{-\nu w_{i\bar{i}} + \tfrac{1}{2} w^i (u_i + u_{\bar{i}}) + \tfrac{1}{2} u'^i (w_i + w_{\bar{i}})\} = -\operatorname{grad} q, \tag{28}$$

as well as $\operatorname{div} w = 0$ and $w|_s = 0$. Multiplying (28) scalarly by w, we get the identity

$$\nu \|w_x\|^2 + A(w, w, u) = 0$$

which leads, due to (15), to the inequality

$$\nu \|w_x\|^2 \leqslant c_0 c_1 \|w_x\|^2 \|u_x\|.$$

The last inequality and (27) imply $\|w_x\| = 0$ and hence $u = u'$ which, in turn, implies $p = p'$ due to (19).

8. A vector $\mathscr{U}$ will be called a *weak solution* of problem (1) if $\mathscr{U} - \mathscr{B} \in \overset{\circ}{J}(\Omega)$, $\|\nabla \mathscr{U}\|$ is finite and if the identity

$$\int_\Omega \nabla \mathscr{U} \cdot \nabla \theta dx + \int_\Omega \sum_i \mathscr{U}^i \frac{\partial \mathscr{U}}{\partial x^i} \theta dx = \int_\Omega \mathscr{T} \theta dx \tag{29}$$

is satisfied for any $\theta \in N$, N denoting the class of regular divergence free vectors defined in Ω and vanishing near S.

From now on we assume that a in (20) does not depend on h.

Let $\{h\}$ be any sequence of positive numbers decreasing to zero. Let $\{u_h\}$ denote the corresponding set of solutions of system (16)-(19). Finally, let U_h denote the approximation of u_h defined in section 2 of [2].

THEOREM 3. *One can extract from* $\{U_h\}$ *a subsequence* $\{U_{h'}\}$ *which converges to a weak solution* $\mathscr{U}$ *of the problem* (1).

Proof. Take $\theta \in N$ and form its solenoidal approximation ϑ provided by formulas (6) (or (4) of [2]). Due to the regularity of θ, ϑ converges uniformly to θ when $h \to 0$. Now a standard device is to be applied. The norms $\|u_x\|$ are, due to (26), uniformly bounded and this allows us to extract a subsequence $\{u_{h'}\}$ such that the extensions $U_{h'}$ converge strongly to some $\mathscr{U}$, $(U_{h'})_x$ weakly to $\nabla\mathscr{U}$ under the $L_2(\Omega)$-norm. Hence, multiplying equation (16) scalarly by ϑ and letting h tend to zero we get (29). The equation $\nabla\cdot\mathscr{U} = 0$ and the condition $\mathscr{U} - \mathscr{B} \in \mathring{J}(\Omega)$ are obviously satisfied.

Remark. If condition (27) is satisfied, then the whole sequence $\{u_h\}$ converges to $\mathscr{U}$.

9. The problem we intend to discuss now is that of approximate solving of system (16)-(19). This will be done in two different ways: either by an iterative process of solving of sets of linear equations or by utilizing the asymptotic behaviour at $t \to \infty$ of finite-difference approximations to the non-stationary solutions of the Navier-Stokes equations, both applicable in 2- and 3-dimensional spaces as well. Either of these procedures is limited to the solutions u subject to the unicity condition (27). Moreover, in the second of those approximating procedures the condition $kh^{-2} \leqslant$ const is to be imposed on the mesh sizes k and h of the grid of t, x space.

THEOREM 4. *Under condition* (27) *of Theorem* 2 *the solution* u *of the system* (16)-(19) *is equal, for any fixed* h, *to* $\lim_m v^{(m)}$ *of the sequence* $v^{(m)}$, $m = 0, 1, \ldots$, *of uniquely determined solutions of the following linear system of equations*:

$$(30)\qquad \begin{gathered} \sum_i \{-\nu v_{i\bar{i}}^{(m)} + \tfrac{1}{2}(v^{(m-1)})^i (v_i^{(m)} + v_{\bar{i}}^{(m)})\} = -\operatorname{grad} p^{(m)} + f, \\ \operatorname{div} v^{(m)} = 0, \qquad v^{(m)}|_S = b|_S, \qquad \sum_{\omega^*} p^{(m)} = 0. \end{gathered}$$

$v^{(0)}$ *is an arbitrary element of* $\mathring{j}(\omega)$. *Equations* (30) *are taken in the points of the same sets of* ω *as it was in the case of the system* (16)-(19).

Proof. Multiplying scalarly the first of equations (30) by $v^{(m)}$ we get

$$\nu \|v_x^{(m)}\|^2 = (f, v^{(m)}),$$

hence $v^{(m)} = 0$ if only $f = b = 0$. The number of equations (30) is equal to the number of unknowns $u^{(m)}, p^{(m)}$ and therefore equations (30) are uniquely solvable for any m.

Subtract, side by side, equations of the first group of (30) from equations (16) and write $u - v^{(m)} = w$, $u - v^{(m-1)} = \overline{w}$, $q = p - p^{(m)}$. The result has the form

$$\sum_i \{-\nu w_{i\bar{i}} + \tfrac{1}{2}\overline{w}^i (u_i + u_{\bar{i}}) + \tfrac{1}{2}(v^{(m-1)})^i (w_i + w_{\bar{i}})\} = -\operatorname{grad} q .$$

Multiplying the last equation scalarly by w, we get

$$\nu \|w_x\|^2 + A(w, \overline{w}, u) = 0 .$$

The result of applying the estimate (14) to A above is

$$(2\nu - c_0 c_1 \|u_x\|) \|w_x\|^2 \leqslant c_0 c_1 \|u_x\| \|\overline{w}_x\|^2 ,$$

which gives us

$$\|w_x\| \leqslant \eta^{m/2} \|v_x^{(1)} - v_x^{(0)}\| ,$$

with $\eta = c_0 c_1 \|u_x\| (2\nu - c_0 c_1 \|u_x\|)^{-1}$. Now condition (27) leads to $0 < \eta < 1$ and hence to

$$\lim_m v^{(m)} = u .$$

Remark. If we reject condition (27), the following may be proved: from the sequence $v^{(m)}$ of solutions of system (30) a subsequence may be chosen which tends to a solution of system (16)-(19).

10. To describe the second approximation mentioned above we could use any of schemes presented in [3]. We can also use the other schemes, for example the following one:

$$\text{(31)} \qquad \begin{aligned} & v_t + \sum_i \{-\nu v_{i\bar{i}} + \tfrac{1}{2}\overline{v}^i (v_i + v_{\bar{i}})\} = -\operatorname{grad} p + f , \\ & \operatorname{div} v = 0 , \qquad \sum_{\omega^*} p = 0 , \qquad v|_s = 0 , \end{aligned}$$

where the same notation has been used as in [2]. The proposed scheme differs slightly only from one of the schemes given in [3]. b is assumed to be zero, f is independent of the time variable. Equations (31) are to be taken in the same points of the set ω as before when considering equations (16)-(19). t-variable assumes the values $t = nk \leqslant T$, k is a positive constant.

The proposed scheme provides, for any initial condition, convergent approximations in any time-interval $(0, T)$ due to the fact that all the procedure of [3] may be applied to the present case. This is a consequence of the possibility of establishing the basic a priori estimate (see formula (9) of [3] or cf. a similar situation in [2]).

THEOREM 5. *Let v be, for fixed h and k and $t = nk \leqslant T = Nk$, a solution of the system* (31) *subject to the initial condition $\bar{v}|_{n=1} = a$ with a resulting by approximation (defined as in* [2] *or* [3]) *of any $\mathscr{A} \in \mathring{J}(\Omega)$. If condition* (27) *is satisfied and the fixed h, k are such that*

$$kh^{-2} \leqslant K, \tag{32}$$

where K is a constant given below in formula (36), *then the solution v of* (31) *converges to the solution u of* (16)-(19), *when $T \to \infty$, in the sense that $v \to u$ in each point of ω.*

Proof. Subtract, side by side, equations (16) from the first group of equations (31) taken at $t = nk$. If we put $w = v - u$ and note that $u_t = w_t$, we may write the result in the following form:

$$w_t + \sum_i \{-\nu w_{i\bar{i}} + \tfrac{1}{2}\bar{w}^i(w_i + w_{\bar{i}}) + \tfrac{1}{2}\bar{w}^i(u_i + u_{\bar{i}})\} = -\operatorname{grad} p',$$

where p' denotes the difference of p's appearing in (31) and (16). Multiplying scalarly the last equation by $2w$, both taken at the same $t = nk$, and then making use of the identity

$$2h^2 \sum (w - \bar{w})w = \|w\|^2 - \|\bar{w}\|^2 + \|w - \bar{w}\|^2,$$

of (12) and of estimate (14), we get the inequality

$$\|w\|^2 - \|\bar{w}\|^2 + \|w - \bar{w}\|^2 + 2\nu k\|w_x\|^2 \leqslant kc_0c_1\|u_x\|(\|w_x\|^2 + \|\bar{w}_x\|^2), \tag{33}$$

valid for any $t = nk \leqslant T$.

Making use of the triangle inequality and of the crude estimate

$$\|w_i - \bar{w}_i\| \leqslant 2h^{-1}\|w - \bar{w}\|,$$

we may write

$$\|\bar{w}_x\|^2 \leqslant (1+\lambda)\|w_x\|^2 + 8(1+\lambda^{-1})h^{-2}\|w - \bar{w}\|^2,$$

where λ is any positive constant. Applying this to (33) we get

$$\begin{aligned}\|w\|^2 - \|\bar{w}\|^2 + \{1 - 8(1+\lambda^{-1})c_0c_1kh^{-2}\|u_x\|\}\|w - \bar{w}\|^2 + \\ + k\{2\nu - (2+\lambda)c_0c_1\|u_x\|\}\|w_x\|^2 \leqslant 0.\end{aligned} \tag{34}$$

Put $\nu - c_0c_1\|u_x\| = \varepsilon$. Due to (27), ε is positive. If we now put $\lambda = \varepsilon(\nu - \varepsilon)^{-1}$ into (34), we get

$$\|w\|^2 - \|\bar{w}\|^2 + k\varepsilon\|w_x\|^2 \leqslant 0, \tag{35}$$

if only

$$kh^{-2} \leqslant K = (\nu - c_0c_1\|u_x\|)(8\nu c_0c_1\|u_x\|)^{-1}. \tag{36}$$

Applying inequality (8) to (35) we infer that

$$\|w\|^2 \leqslant (1+k\varepsilon c_0^2)^{-1} \|\overline{w}\|^2,$$

whence

$$\|w(N)\| < (1+k\varepsilon c_0^2)^{-N/2} \|u-a\|$$

which, due to the definition of w, implies $v \to u$ with $N \to \infty$, i.e., with $T \to \infty$, k being fixed. For large $Nk = T$ the coefficient in the last inequality is approximately equal to $\exp(-\frac{1}{2}T\varepsilon c_0^2)$.

REFERENCES

[1] H. Fujita, *On the existence and regularity of the steady-state solutions of the Navier-Stokes equations*, Journal of the Faculty of Sciences, University of Tokyo, 9 (1961-1963), p. 59-102.

[2] A. Krzywicki, *On a new finite-difference scheme for the non-stationary Navier-Stokes equations*, Colloquium Mathematicum, this fascicule, p. 143-155.

[3] А. Кживицки и О. А. Ладыженская, *Метод сеток для нестационарных уравнений Навье-Стокса*, Труды Математического Института им. В. А. Стеклова 92 (1966), p. 93-99.

[4] О. А. Ладыженская, *Математические вопросы динамики вязкой несжимаемой жидкости*, Москва 1961.

INSTITUTE OF MATHEMATICS OF THE WROCŁAW UNIVERSITY

Reçu par la Rédaction le 1. 4. 1967