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A THEOREM ON JOINT PROBABILITY DISTRIBUTIONS IN STOCHASTIC LOCALLY CONVEX LINEAR TOPOLOGICAL SPACES

BY

I. KOTLARSKI (WARSZAWA)

Let $(\Omega, \mathscr{B}, P)$ be a probability space, $\mathscr{X}$ a real locally convex linear topological space and $X(\omega)$ a strongly $\mathscr{B}$-measurable mapping from Ω into $\mathscr{X}$ which will be also called a *random variable* in $\mathscr{X}$ (random $\mathscr{X}$-variable). Its distribution $P_X(A) = P(\omega\colon X(\omega) \in A \subset \mathscr{X})$ is uniquely determined by the characteristic functional

$$f_X(x^*) = Ee^{ix^*(X)} = \int_{\Omega} e^{ix^*(x)} P(d\omega). \tag{1}$$

Throughout this paper x (may be with indices) belongs to $\mathscr{X}$, x^* belongs to $\mathscr{X}^*$, where $\mathscr{X}^*$ is the space conjugate to $\mathscr{X}$, and $k = 1, 2, 3$.

It is well known that the characteristic functional of a random $\mathscr{X}$-variable has some properties similar to those of the characteristic function of a real random variable (see [1]). They will be used in the sequel. The aim of this paper is to prove the following

THEOREM. *Let X_k $(k = 1, 2, 3)$ be independent random $\mathscr{X}$-variables and let*

$$Y_1 \overset{\text{df}}{=} X_1 + X_3, \qquad Y_2 \overset{\text{df}}{=} X_2 + X_3. \tag{2}$$

If the joint characteristic functional of (Y_1, Y_2) does not vanish, then it determines all distributions of X_k up to a change of location.

Proof. Denote the characteristic functional of X_k by $f_k(x^*)$ and the joint characteristic functional of (Y_1, Y_2) by

$$f(x_1^*, x_2^*) = E\{e^{i[x_1^*(Y_1) + x_2^*(Y_2)]}\}. \tag{3}$$

It is easy to see that

$$f(x_1^*, x_2^*) = f_1(x_1^*)\cdot f_2(x_2^*)\cdot f_3(x_1^* + x_2^*). \tag{4}$$

If X'_k are other independent random $\mathscr{X}$-variables such that the joint characteristic functional $f'(x_1^*, x_2^*)$ of $(Y'_1, Y'_2) = (X'_1 + X'_3, X'_2 + X'_3)$ does not vanish, then also

$$f'(x_1^*, x_2^*) = f'_1(x_1^*) \cdot f'_2(x_2^*) \cdot f'_3(x_1^* + x_2^*), \tag{5}$$

where $f'_k \equiv f_{X'_k}$.

Now we assume that

$$f'(x_1^*, x_2^*) = f(x_1^*, x_2^*). \tag{6}$$

By this assumption and from (4) and (5) we obtain the equation

$$f'_1(x_1^*) f'_2(x_2^*) f'_3(x_1^* + x_2^*) = f_1(x_1^*) f_2(x_2^*) f_3(x_1^* + x_2^*). \tag{7}$$

Let us put

$$f'_k(x^*) = f_k(x^*) \cdot g_k(x^*), \qquad k = 1, 2, 3. \tag{8}$$

Since the left-hand sides of (4) and (5) do not vanish, it follows that also none of the functionals f'_k, f_k and g_k does. Putting (8) into (7) we obtain for the unknown functionals g_k the equation

$$g_1(x_1^*) g_2(x_2^*) g_3(x_1^* + x_2^*) = 1. \tag{9}$$

These functionals are continuous in the weak* topology of $\mathscr{X}^*$ and satisfy the conditions $g_k(0) = 1$ and $g_k(-x^*) = \overline{g_k(x^*)}$. This follows from (8) and the same properties of characteristic functionals f_k and f'_k (see [1]). Putting $x_1^* = x^*, x_2^* = 0$ in (9), and then $x_1^* = 0$, $x_2^* = x^*$, we obtain

$$g_1(x^*) = g_2(x^*) = 1/g_3(x^*). \tag{10}$$

Putting (10) into (9) we obtain for g_3 the equation

$$g_3(x_1^* + x_2^*) = g_3(x_1^*) \cdot g_3(x_2^*). \tag{11}$$

Let now

$$h(x^*) = h_1(x^*) + i h_2(x^*) = \ln g_3(x^*), \tag{12}$$

where ln is the continuous branch of the logarithm which satisfies the conditions

$$h(0) = h_1(0) + i h_2(0) = \ln g_3(0) = \ln 1 = 0, \tag{13}$$

and $h_j(x^*)$, $j = 1, 2$, are real. Then equation (11) takes on the form

$$h_j(x_1^* + x_2^*) = h_j(x_1^*) + h_j(x_2^*), \qquad j = 1, 2, \tag{14}$$

where $h_j(x^*)$ are real weakly* continuous functionals satisfying conditions (13) and

$$h_1(-x^*) = h_1(x^*), \qquad h_2(-x^*) = -h_2(x^*). \tag{15}$$

Then we see that $h_1(x^*) \equiv 0$ and $h_2(x^*)$ is a real linear functional on $\mathscr{X}^*$ continuous in the weak* topology. By the Banach's theorem (see [2], p. 112) h_2 is of the form $h_2(x^*) = x^*(x_0)$ with a certain fixed $x_0 \in \mathscr{X}$. Using (10) and (12) we obtain that

$$g_3(x^*) = e^{ix^*(x_0)} \quad \text{and} \quad g_1(x^*) = g_2(x^*) = e^{-ix^*(x_0)}. \tag{16}$$

Finally, putting (16) into (8), we see that

$$f_j'(x^*) = f_j(x^*)\cdot e^{-ix^*(x_0)},\ j = 1, 2, \quad \text{and} \quad f_3'(x^*) = f_3(x^*)\cdot e^{ix^*(x_0)}, \tag{17}$$

where x_0 is a fixed element of $\mathscr{X}$. This ends the proof.

Remark. An analogous theorem can be proved in a similar way (using Pontryagin's duality theorem instead of Banach's theorem) if $\mathscr{X}$ is a locally compact abelian topological group.

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