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PSEUDO-SYMMETRIC DIFFERENTIATION

BY

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Let f be a real-valued function defined on the real line R. If φ and ψ are positive, continuous, strictly increasing functions defined on an interval $(0, \delta)$ such that the right-hand limits $\varphi(0+) = \psi(0+) = 0$, then we consider the quotient

$$q(x, h) = \frac{f(x+\varphi(h))-f(x-\psi(h))}{\sigma(h)},$$

where $\sigma(h) = \varphi(h)+\psi(h)$ for $h \in (0, \delta)$, and we define the upper, lower $\varphi\psi$-pseudo-symmetric derivates, the $\varphi\psi$-pseudo-symmetric derivative of f at x in the usual manner, namely

$$\bar{f}_{\varphi\psi}(x) = \overline{\lim_{h\to 0+}}\, q(x, h), \qquad \underline{f}_{\varphi\psi}(x) = \underline{\lim_{h\to 0+}}\, q(x, h),$$

and, when the limit exists,

$$f'_{\varphi\psi}(x) = \lim_{h\to 0+} q(x, h).$$

This derivative is more general than that considered in [6]. In the present paper it is proved that if $f'_{\varphi\psi}(x)$ exists at every $x \in R$, then $f'_{\varphi\psi}$ is of Baire class 1. Also, with some further restriction on φ and ψ, a certain monotone theorem is proved and, finally, it is shown that $f'_{\varphi\psi}$ has the Denjoy property if it has the Darboux property.

1. In this section, f, φ, ψ, etc. are as stated above.

THEOREM 1. *If $f'_{\varphi\psi}(x)$ exists for every $x \in R$, then $f'_{\varphi\psi}$ is of Baire class* 1.

Proof. Let $a < d$ and let a perfect set $P \subset R$ be given. According to Preiss ([4], Theorem 1), it suffices to show that

$$A = \{x \in P: f'_{\varphi\psi}(x) \leqslant a\} \quad \text{and} \quad D = \{x \in P: f'_{\varphi\psi}(x) \geqslant d\}$$

cannot both be dense in P. We assume that both A and D are dense in P and want to arrive at a contradiction.

Let b, c be real numbers such that $a < b < c < d$ and let

$$B = \{x \in P: f'_{\varphi\psi}(x) > b\}, \qquad C = \{x \in P: f'_{\varphi\psi}(x) < c\}.$$

Then $P = B \cup C$ and at least one of B, C is of the second category in P. However, we shall show that B cannot be of the second category in P and a similar argument (D and C are used instead of A and B, respectively) shows that C cannot be of the second category in P. Thus the theorem will be proved.

Suppose that B is of the second category in P. With no loss of generality, we assume that $b = 0$ (we can consider $f(x)-bx$ if $b \neq 0$). Let n_0 be the first positive integer such that $n_0^{-1} < \delta$. For $n \geqslant n_0$, we set

$$B_n = \{x \in P\colon f(x+\varphi(h)) > f(x-\psi(h)) \text{ for } h \in (0, 1/n)\}. \tag{1}$$

Then $\{B_n\}$ is an expanding sequence of sets with

$$B \subset \bigcup_{n=n_0}^{\infty} B_n.$$

There must be some $n_1 \geqslant n_0$ such that B_{n_1} is not nowhere dense in P. That is, there exists a closed interval I whose interior I° contains points of P, and $I \cap P$ is contained in $\bar{B}_{n_1}$, the closure of B_{n_1}. We can assume that $I \cap P$ is perfect. (Otherwise, consider a suitable subinterval of I.)

Since the function σ is clearly continuous and strictly increasing on $(0, \delta)$ with a limit 0 at 0 and $1/n_1 < \delta$, there exists $r \in (0, \delta)$ such that $\sigma(r) = \frac{1}{2}\sigma(1/n_1)$. For each $x \in A$, since $f'_{\varphi\psi}(x) \leqslant a < b = 0$, there exists $\delta_x \in (0, r)$ such that

$$f(x+\varphi(h))-f(x-\psi(h)) < \sigma(h)\frac{a}{2} < 0 \tag{2}$$

for every $h \in (0, \delta_x)$.

Now $I^\circ \cap P \neq \emptyset$ and A being dense in P, there exists

$$x_0 \in A \cap I^\circ \cap P.$$

Since $I \cap P$ is perfect, x_0 is either a right or a left limit point of $I \cap P$. The two cases are analogous, and hence we give only the proof for the case in which x_0 is a right limit point of $I \cap P$.

We fix a point

$$y \in A \cap (x_0, x_0 + \min\{\varphi(\delta_{x_0}), \psi(\delta_{x_0})\}) \cap I \cap P.$$

Clearly,

$$0 < y - x_0 < \min\{\varphi(\delta_{x_0}), \psi(\delta_{x_0})\} \leqslant \psi(\delta_{x_0})$$

and $\psi^{-1}(y-x_0)$ is defined. Let

$$H = (0, \min\{\delta_y, \psi^{-1}(y-x_0)\}).$$

Then, for $h \in H$, $0 < \psi(h) < y - x_0$, and hence

$$0 < y - x_0 - \psi(h) < y - x_0 < \min\{\varphi(\delta_{x_0}), \psi(\delta_{x_0})\} \leqslant \varphi(\delta_{x_0}).$$

We can define a map t on H by setting, for $h \in H$,

$$t_h = \varphi^{-1}\big(y - x_0 - \psi(h)\big). \tag{3}$$

For $h \in H$, the above inequalities show that

$$0 < t_h < \delta_{x_0}. \tag{4}$$

Also,

$$\begin{aligned} y + \varphi(h) - x_0 + \psi(t_h) &= y + \varphi(h) - x_0 + \sigma(t_h) - \varphi(t_h) \\ &= y + \varphi(h) - x_0 + \sigma(t_h) - \big(y - x_0 - \psi(h)\big) = \sigma(t_h) + \sigma(h). \end{aligned} \tag{5}$$

By (4), the definition of H, and the choice of δ_x for $x \in A$, both t_h and h are in the interval $(0, r)$, and hence both $\sigma(t_h)$ and $\sigma(h)$ are positive and less than $\sigma(r) = \frac{1}{2}\sigma(1/n_1)$. Thus (5) implies

$$0 < y + \varphi(h) - x_0 + \psi(t_h) < \sigma(1/n_1).$$

Now we can define another map λ on H by setting, for $h \in H$,

$$\lambda_h = \sigma^{-1}\big(y + \varphi(h) - x_0 + \psi(t_h)\big) \tag{6}$$

and assert that

$$0 < \lambda_h < 1/n_1 \quad \text{for } h \in H. \tag{7}$$

Further we define a map z on H by

$$z_h = y + \varphi(h) - \varphi(\lambda_h) \quad \text{for } h \in H. \tag{8}$$

It should be noted that these maps are continuous on the interval H. We assert that $z(H)$ is an interval with x_0 as its left-hand endpoint. This is shown by (9) and (11) below.

For $h \in H$, by (6), (5), and the fact that $\sigma(0+) = 0$, we have

$$\lim_{h \to 0+} \sigma(\lambda_h) = \lim_{h \to 0+} \sigma(t_h).$$

Since σ^{-1} and φ^{-1} are clearly continuous, we see from the above equality and (3) that

$$\lim_{h \to 0+} \lambda_h = \lim_{h \to 0+} t_h = \varphi^{-1}(y - x_0).$$

Thus $\lim_{h \to 0+} \varphi(\lambda_h) = y - x_0$ and, by (8),

$$\lim_{h \to 0+} z_h = x_0.$$

That is,

$$x_0 \in \overline{z(H)}. \tag{9}$$

Also,

$$z_h = y+\varphi(h)-\varphi(\lambda_h) = y+\varphi(h)-\sigma(\lambda_h)+\psi(\lambda_h).$$

This together with (6) yields

$$(10) \qquad z_h = x_0-\psi(t_h)+\psi(\lambda_h).$$

By (6) and (5), $\sigma(\lambda_h) = \sigma(t_h)+\sigma(h) > \sigma(t_h)$, and hence $\lambda_h > t_h$. It follows that $\psi(\lambda_h) > \psi(t_h)$ and, from (10),

$$(11) \qquad z_h > x_0.$$

Since x_0 is a right limit point of $I \cap P$ and $\bar{B}_{n_1} \supset I \cap P$, the set $z(H) \cap I \cap P$ is not empty and contains points of B_{n_1}. Let

$$\xi \in B_{n_1} \cap z(H) \cap I \cap P.$$

Then there exists $h \in H$ such that

$$(12) \qquad \xi = z_h = y+\varphi(h)-\varphi(\lambda_h).$$

For this ξ and h, we have $\xi+\varphi(\lambda_h) = y+\varphi(h)$. Also, by (12), (10), and (3),

$$\xi-\psi(\lambda_h) = x_0-\psi(t_h) \quad \text{and} \quad x_0+\varphi(t_h) = y-\psi(h).$$

Thus we have

$$(13) \quad f(x_0+\varphi(t_h))-f(x_0-\psi(t_h))$$
$$= -[f(y+\varphi(h))-f(y-\psi(h))]+[f(\xi+\varphi(\lambda_h))-f(\xi-\psi(\lambda_h))].$$

Since $x_0 \in A$, (2) and (4) imply that the left-hand side of (13) is less than zero. But $y \in A$ and $\xi \in B_{n_1}$; we see from the fact that $h \in H$, (2), (7), and (1) that the right-hand side of (13) is greater than zero. This is a contradiction.

2. In this section we assume that f is a measurable function and that φ and ψ satisfy a further condition:

There exist positive numbers η and M such that, for each $h \in (0, \delta)$,

$$\eta \leqslant \underline{\varphi}_+(h) \leqslant \bar{\varphi}^+(h) \leqslant M \quad \text{and} \quad \eta \leqslant \underline{\psi}_+(h) \leqslant \bar{\psi}^+(h) \leqslant M,$$

where $\underline{\varphi}_+(h)$ and $\bar{\varphi}^+(h)$ denote the lower and upper right derivates of φ at h, respectively.

THEOREM 2. *A finite derivative $f'(x)$ exists almost everywhere on the set*

$$\{x: \bar{f}_{\varphi\psi}(x) < +\infty\} \cup \{x: \underline{f}_{\varphi\psi}(x) > -\infty\}.$$

This theorem is proved in [5].

THEOREM 3. *If $\underline{f}_{\varphi\psi}(x) > -\infty$ on R and $\underline{f}_{\varphi\psi}(x) \geqslant 0$ almost everywhere on R, then the restriction $f|C$ of f on C is non-decreasing, where C is the set of points of continuity of f.*

To prove this theorem, we need the following

LEMMA. *If* $\underline{f}_{\varphi\psi}(x) \geqslant 0$ *on* R, *then the restriction* $f|C$ *is non-decreasing.*

Proof. We assume that $\underline{f}_{\varphi\psi}(x) > 0$ for every $x \in R$. The general case follows by considering $f(x)+\varepsilon x$ for each $\varepsilon > 0$.

Suppose that there exist a, b in C with $a < b$ and $f(a) > f(b)$. By Theorem 2, C is dense. Let

$$x_0 \in (a, b) \cap C$$

and let $\alpha \in (f(b), f(a)) - \{f(x_0)\}$ be fixed. Then at least one of the sets

$$E_\alpha = \{x \in [a, b]: f(x) \geqslant \alpha\} \quad \text{and} \quad E^\alpha = \{x \in [a, b]: f(x) \leqslant \alpha\}$$

contains a subinterval of (a, b). We assume that there exists $(c, d) \subset (a, b)$ such that $(c, d) \subset E^\alpha$. (The case where $(c, d) \subset E_\alpha$ can be proved analogously.) Let

$$c_0 = \inf\{x \in [a, b]: (x, d) - E^\alpha \text{ is countable}\}.$$

Clearly, $a \leqslant c_0 \leqslant c$ and $(c_0, d) - E^\alpha$ is countable. If $c_0 > a$, then, since $\underline{f}_{\varphi\psi}(c_0) > 0$, there exists $\delta' \in (0, \delta)$ such that

$$\varphi(\delta') < d - c_0, \quad \psi(\delta') < c_0 - a,$$

and

$$f(c_0 - \psi(h)) < f(c_0 + \varphi(h)) \quad \text{for } h \in (0, \delta').$$

It follows from the above inequalities that $c_0 - \psi(\delta') \in (a, c_0)$, and $(c_0 - \psi(\delta'), c_0) - E^\alpha$ is countable. Hence $(c_0 - \psi(\delta'), d) - E^\alpha$ is countable. This is a contradiction to the definition of c_0. Thus we must have $c_0 = a$. Since $a \in C$, f is continuous at c_0 and

$$f(c_0) = \lim_{\substack{x \to c_0+ \\ x \in E^\alpha}} f(x) \leqslant \alpha.$$

On the other hand, $f(c_0) = f(a) > \alpha$. We arrive at a contradiction. The Lemma is proved.

Proof of Theorem 3. Let $E = \{x: \underline{f}_{\varphi\psi}(x) < 0\}$. Then $|E|$, the Lebesgue measure of E, is zero. By a theorem on p. 214 in [3], there exists a non-decreasing continuous function g such that $g'(x) = +\infty$ for every $x \in E$. Let $\varepsilon > 0$ and $F(x) = f(x) + \varepsilon g(x)$ for $x \in R$. Then $\underline{F}_{\varphi\psi}(x) \geqslant 0$ on R. Since the set of points of continuity of F is the same as that of f, namely C, by the Lemma, $F|C$ is non-decreasing. This holds for every $\varepsilon > 0$. Thus Theorem 3 is proved.

COROLLARY. *Let* $f'_{\varphi\psi}(x)$ *exist and* $f'_{\varphi\psi}(x) \neq -\infty$ *on* R. *If* $f'_{\varphi\psi}(x) \geqslant 0$ *almost everywhere on* R, *then* $f'_{\varphi\psi}(x) \geqslant 0$ *on* R.

Proof. Suppose that there exists $x_0 \in R$ such that $f'_{\varphi\psi}(x_0) < 0$. Then there exists $\delta' \in (0, \delta)$ such that

$$f(x_0+\varphi(h)) < f(x_0-\psi(h)) \quad \text{for } h \in (0, \delta'). \tag{14}$$

Since C is a G_δ-set and, by Theorem 2, C is dense, C is residual in R. Noting that φ and ψ are actually homeomorphisms, we see that the sets

$$\{h \in (0, \delta'): x_0-\psi(h) \in C\}, \quad \{h \in (0, \delta'): x_0+\varphi(h) \in C\}$$

are both residual in $(0, \delta')$. There exists $h_0 \in (0, \delta')$ such that $x_0-\psi(h_0)$ and $x_0+\varphi(h_0)$ are points in C. By Theorem 3,

$$f(x_0-\psi(h_0)) \leqslant f(x_0+\varphi(h_0)).$$

This contradicts (14). The proof is completed.

Remark 1. The above Corollary remains valid if R is replaced by any open interval. It clearly follows that, for f with finite $f'_{\varphi\psi}(x)$ on I, if $f'_{\varphi\psi}(x) \geqslant M$ (or $f'_{\varphi\psi}(x) \leqslant m$) almost everywhere on I, then the same inequality holds everywhere on I, where I is an open interval.

THEOREM 4. *Let f have a finite $f'_{\varphi\psi}(x)$ on R. If $f'_{\varphi\psi}$ has the Darboux property, then for given intervals (a, b) and (α, β) the set*

$$\{x \in (a, b): f'_{\varphi\psi}(x) \in (\alpha, \beta)\}$$

is either empty or of positive Lebesgue measure.

This theorem can be proved in a standard way by using Remark 1 and Theorem 1.

Remark 2. Applying Theorem 4 and Darboux property of $f'_{\varphi\psi}$, we can easily show that, in the statement of Theorem 4, (a, b) can be replaced by $[a, b]$.

Remark 3. The additional condition imposed on φ and ψ at the beginning of Section 2 is directly used only for the proof of Theorem 2. For the rest of the results in this section, this condition is required simply because Theorem 2 is applied. Therefore, this condition can be replaced by any other condition under which Theorem 2 holds.

Remark 4. Theorem 1 is a direct generalization of a theorem in [1]. Also, in his paper [2], Larson introduced generalized approximate parametric derivatives which include the $\varphi\psi$-pseudo-symmetric derivative if a further restriction on φ, ψ as mentioned at the beginning of Section 2 is imposed. There he proved that Theorem 1 above holds for his generalized derivatives if f is measurable.

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