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THE SPECTRAL DECOMPOSITION OF WEIGHTED SHIFTS AND THE A_p CONDITION

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1. Introduction. The Hilbert transform has had a substantial influence on many developments in analysis; the aim of this paper is to bring together two areas in which it has played a rôle, one from harmonic analysis and the other from Banach space operator theory. The harmonic analysis in question is the A_p condition of B. Muckenhoupt (or more precisely the discrete version of this condition). This was introduced originally in [8] to give a condition on a weight function under which the Hardy–Littlewood maximal operator satisfies a weighted norm inequality and was developed further in [7], where the corresponding inequality for the Hilbert transform is studied. On the other hand, the relevant operator theory is concerned with the possibility of representing an operator V on a Banach space X as

$$V = \int_{0^-}^{2\pi} e^{i\lambda}\, dE(\lambda),$$

the function $E(\cdot)$ being projection-valued (with various additional properties to be specified later) and the integral existing in the strong operator topology. Such operators are called *trigonometrically well-bounded* and were introduced in [1]. The main result presented here is that the bilateral shift on a discrete weighted l^p space ($1 < p < \infty$) is trigonometrically well-bounded if and only if the A_p condition holds for the corresponding sequence of weights (Theorem (4.2)). As an application, we show that trigonometrical well-boundedness for an invertible operator V on a Hilbert space cannot be characterized by the growth rate of $\|V^n\|$ as $|n| \to \infty$. This contrasts with the result that, on a Hilbert space, an invertible operator V is of scalar type with spectrum contained in the unit circle if and only if $\|V^n\| = O(1)$ as $|n| \to \infty$.

As usual, $\mathbf{R}$, $\mathbf{C}$, $\mathbf{Z}$ and $\mathbf{N}$ denote the real numbers, the complex numbers, the integers and the positive integers respectively, and $\mathbf{T}$ is the circle group. Given a Banach space X, $B(X)$ denotes the algebra of all bounded linear operators on X and I is the identity operator on X. We denote by $\mathrm{BV}[0,2\pi]$ the Banach algebra of complex-valued functions of bounded variation on $[0,2\pi]$, with norm

$$\|f\|_{[0,2\pi]} = |f(2\pi)| + \operatorname{var} f,$$

where $\operatorname{var} f$ is the total variation of f on $[0,2\pi]$, and $\mathrm{AC}[0,2\pi]$ is the subalgebra of absolutely continuous functions on $[0,2\pi]$. Further, $\mathrm{BV}(\mathbf{T})$ is the Banach algebra of functions f on $\mathbf{T}$ such that the associated function $\tilde{f}(t) = f(e^{it})$ belongs to $\mathrm{BV}[0,2\pi]$, with $\|f\|_{\mathbf{T}} = \|\tilde{f}\|_{[0,2\pi]}$. The kth Fourier coefficient of an integrable function f on $[0,2\pi]$ or $\mathbf{T}$ is denoted by $\hat{f}(k)$.

By a *weight sequence* (or just a *weight*), we shall mean a sequence $w = \{w_k\}_{k\in\mathbf{Z}}$ of strictly positive real numbers. For $1 \le p < \infty$ and a weight w, $l^p(w)$ denotes the usual weighted l^p space of complex sequences $x = \{x_k\}_{k\in\mathbf{Z}}$ such that

$$\|x\|_{w,p} \equiv \left\{\sum_{k\in\mathbf{Z}} |x_k|^p w_k\right\}^{1/p} < \infty.$$

Given any complex sequence $x = \{x_k\}_{k\in\mathbf{Z}}$, let Ux denote the sequence $Ux = \{x_{k-1}\}_{k\in\mathbf{Z}}$, so that U denotes the bilateral right shift on any sequence space on which it is defined. It is easy to verify that U is a bounded linear mapping of $l^p(w)$ into itself if and only if

$$\sup\{w_{k+1}/w_k : k \in \mathbf{Z}\} < \infty. \tag{1.1}$$

As a consequence, U is an invertible element of $B(l^p(w))$ if and only if

$$\sup\{w_{k+1}/w_k : k \in \mathbf{Z}\} < \infty \quad \text{and} \quad \sup\{w_k/w_{k+1} : k \in \mathbf{Z}\} < \infty. \tag{1.2}$$

Furthermore, when (1.2) holds, U^{-1} is the left shift on $l^p(w)$ and the norm of U^n is given by

$$\|U^n\| = \sup\{(w_{k+n}/w_k)^{1/p} : k \in \mathbf{Z}\} \tag{1.3}$$

for $n \in \mathbf{Z}$. The space of bilateral complex sequences having only finitely many non-zero terms is denoted by l_0.

2. The A_p condition. Throughout this section, let $w = \{w_k\}_{k\in\mathbf{Z}}$ be a sequence of positive weights. For p in the range $1 < p < \infty$, w is said to satisfy the A_p *condition* if there exists a constant C_p such that

$$\left(\sum_{k\in I} w_k\right)\left(\sum_{k\in I} w_k^{-1/(p-1)}\right)^{p-1} \le C_p |I|^p \tag{2.1}$$

for every finite interval I in $\mathbf{Z}$, where $|I|$ denotes the cardinality of I. The continuous variable version of this condition was originally introduced by B. Muckenhoupt [8] and has featured extensively in recent years in the study of weighted norm inequalities for the classical operators of Fourier analysis (see [6] for further details). It is the connection between the A_p condition and the Hilbert transform, as discussed in both the continuous and discrete cases in [7], which will concern us here.

Given a sequence $x \in l_0$, let Hx be the sequence defined by

$$(2.2) \qquad (Hx)_k = \sum_{m\in\mathbf{Z}}{}' x_{k-m}/m \qquad \text{for } k \in \mathbf{Z},$$

the prime superscript denoting the omission of the term $m = 0$ in the summation. Thus $Hx = h * x$, where h is the discrete Hilbert kernel given by

$$h(k) = k^{-1} \quad (k \neq 0), \qquad h(0) = 0.$$

For $n \in \mathbf{N}$, let h_n denote the nth truncate of h,

$$h_n(k) = h(k) \quad (|k| \leq n), \qquad h_n(k) = 0 \quad (|k| > n),$$

and let $H^{\#}$ be the maximal function for the sequence of convolution operators $x \to h_n * x$, that is,

$$(H^{\#}x)_k = \sup\Big\{\Big|\sum_{m=-n}^{n}{}' x_{k-m}/m\Big| : n \in \mathbf{N}\Big\}$$

for $k \in \mathbf{Z}$. The definition of $H^{\#}x$ makes sense for an arbitrary complex sequence $x = \{x_k\}_{k\in\mathbf{Z}}$. The main result which will be needed in analyzing the spectral structure of weighted shift operators can now be stated.

(2.3) THEOREM ([7, Theorem 10]). *Let $1 < p < \infty$. Then the following statements are equivalent for the weight sequence w.*

(i) *w satisfies the A_p condition.*

(ii) *There is a constant K_p such that $\|Hx\|_{w,p} \leq K_p\|x\|_{w,p}$ for all sequences $x \in l_0$.*

(iii) *There is a constant K_p such that $\|H^{\#}x\|_{w,p} \leq K_p\|x\|_{w,p}$ for all complex sequences x.*

Remarks. Suppose that (2.3)(ii) holds. It is then easy to see that, for every $k \in \mathbf{Z}$ and every $x \in l^p(w)$, the series in (2.2) converges absolutely, and that (2.2) defines H as a bounded linear mapping of $l^p(w)$ into itself. This is the way in which the above condition (2.3)(ii) is stated in [7]. Further, (2.3)(iii) is formulated in [7] in terms of the maximal operator associated with the tails $h - h_n$ of the discrete Hilbert kernel; the equivalent formulation given here is more convenient for our purposes.

It will be useful to have several simple techniques by which to check the validity of the A_p condition. In order to state these succinctly, we introduce the following terminology. Given an interval J of successive integers (e.g. $\mathbf{N}$ or $-\mathbf{N}$), say that w *satisfies the A_p condition on J* if there is a constant C_p such that (2.1) holds for all finite subintervals I of J.

(2.4) PROPOSITION. *Let $1 < p < \infty$ and let w be a weight sequence on $\mathbb{Z}$.*

(i) *w satisfies the A_p condition on $\mathbb{Z}$ if and only if w satisfies the A_p condition on $\mathbb{N}$ and on $-\mathbb{N}$, and there is a constant C_p such that* (2.1) *holds for every interval I of the form $I = \{k \in \mathbb{Z} : |k| \leq n\}$, where $n \in \mathbb{N}$.*

(ii) *If $w_k = w_{-k}$ for $k \in \mathbb{N}$, then w satisfies the A_p condition on $\mathbb{Z}$ if and only if it satisfies it on $\mathbb{N}$.*

(iii) *If w is bounded on $-\mathbb{N}$ and $w_k \to \infty$ as $k \to \infty$, then w does not satisfy the A_p condition on $\mathbb{Z}$.*

(iv) *Let f be a monotonic, strictly positive function defined on $[1, \infty)$ and let $w_k = f(k)$ for $k \in \mathbb{N}$. Then w satisfies the A_p condition on $\mathbb{N}$ if and only if there is a constant C_p such that*

$$(2.5)\qquad \Big(\int_s^t f(x)\, dx \Big) \Big(\int_s^t [f(x)]^{-1/(p-1)}\, dx \Big)^{p-1} \leq C_p (t-s)^p$$

for all $s, t \in \mathbb{R}$ with $1 \leq s < t < \infty$.

Proof. Throughout, the symbol C_p will be used to denote a constant, the value of which may vary from context to context, but which will only depend on p and w.

(i) Let w satisfy the A_p condition on $\mathbb{N}$ and $-\mathbb{N}$, and suppose that (2.1) holds for every symmetric interval I in $\mathbb{Z}$ as described in the statement of (2.4)(i). Let n and l be non-negative integers and let $m = \max\{n, l\}$. Then

$$\begin{aligned}\Big(\sum_{k=-n}^{l} w_k \Big) \Big(\sum_{k=-n}^{l} w_k^{-1/(p-1)} \Big)^{p-1} &\leq \Big(\sum_{k=-m}^{m} w_k \Big) \Big(\sum_{k=-m}^{m} w_k^{-1/(p-1)} \Big)^{p-1} \\ &\leq C_p (2m+1)^p \leq 2^p C_p (n+l+1)^p.\end{aligned}$$

It follows that w satisfies the A_p condition on $\mathbb{Z}$.

(ii) Suppose that $w_k = w_{-k}$ for $k \in \mathbb{N}$, and that w satisfies the A_p condition on $\mathbb{N}$. Then w also satisfies the A_p condition on $-\mathbb{N}$ and thus, by (i), we need only establish (2.1) for intervals of the form $I_n = \{k \in \mathbb{Z} : |k| \leq n\}$, where $n \in \mathbb{N}$. Fix such an interval. The A_p condition on $\mathbb{N}$ implies that

$$(2.6)\qquad \sum_{k=1}^{n} w_k \leq C_p n^p w_1 \qquad \text{and} \qquad \Big(\sum_{k=1}^{n} w_k^{-1/(p-1)} \Big)^{p-1} \leq C_p n^p w_1^{-1}.$$

The symmetry of w about 0 gives

$$\begin{aligned}(2.7)\qquad &\Big(\sum_{k=-n}^{n} w_k \Big) \Big(\sum_{k=-n}^{n} w_k^{-1/(p-1)} \Big)^{p-1} \\ &= \Big(w_0 + 2 \sum_{k=1}^{n} w_k \Big) \Big(w_0^{-1/(p-1)} + 2 \sum_{k=1}^{n} w_k^{-1/(p-1)} \Big)^{p-1}.\end{aligned}$$

Since $(a+b)^{p-1} \leq 2^{p-1}(a^{p-1}+b^{p-1})$ for $a, b > 0$, we can apply (2.6) and the A_p condition on $\mathbf{N}$ to (2.7) to obtain (2.1) for $I = I_n$ as required.

(iii) Suppose that $w_k \leq M$ for $k \in -\mathbf{N}$ and that $w_k \to \infty$ as $k \to \infty$. For $n \in \mathbf{N}$,

$$n^{-1}\Big(\sum_{k=-n}^{n} w_k\Big) > n^{-1}\Big(\sum_{k=1}^{n} w_k\Big) \to \infty, \tag{2.8}$$

$$n^{-1}\Big(\sum_{k=-n}^{n} w_k^{-1/(p-1)}\Big) > n^{-1}\Big(\sum_{k=-n}^{-1} w_k^{-1/(p-1)}\Big) \geq M^{-1/(p-1)} > 0. \tag{2.9}$$

It follows from (2.8) and (2.9) that

$$(2n+1)^{-p}\Big(\sum_{k=-n}^{n} w_k\Big)\Big(\sum_{k=-n}^{n} w_k^{-1/(p-1)}\Big)^{p-1} \to \infty \quad \text{as} \quad n \to \infty.$$

Hence w does not satisfy the A_p condition on $\mathbf{Z}$.

(iv) Suppose firstly that f is increasing and that (2.5) holds. Let $n, l \in \mathbf{N}$ with $2 \leq n < l < \infty$. Then

$$\sum_{k=n}^{l} w_k \leq \int_{n}^{l+1} f(x)\,dx, \qquad \sum_{k=n}^{l} w_k^{-1/(p-1)} \leq \int_{n-1}^{l} [f(x)]^{-1/(p-1)}\,dx.$$

It follows easily from these inequalities and (2.5) that w satisfies the A_p condition on $\mathbf{N}\setminus\{1\}$. Hence w satisfies the A_p condition on $\mathbf{N}$ (cf. the argument in the proof of (ii) involving (2.6) and (2.7)).

Conversely, suppose that w satisfies the A_p condition on $\mathbf{N}$. Similar arguments to those above show that (2.5) is valid for $s, t \in \mathbf{N}$ with $s < t$. It can then be shown that (2.5) will be valid provided $t - s \geq 1$. If $t - s < 1$, then the left-hand side of (2.5) is dominated by $(t-s)^p w_{k+2} w_k^{-1}$, where $k \in \mathbf{N}$ satisfies $k \leq s < t < k+2$. Since $w_{k+2}w_k^{-1} \leq 3^p C_p$, it is now seen that (2.5) holds for all s, t with $1 \leq s < t < \infty$.

Finally, the case when f is decreasing can be proved similarly, or else can be deduced from the increasing case by considering $f^{-1/(p-1)}$ and p', where p' is the index conjugate to p.

Remark. In fact, we shall only use the sufficiency of (2.5) for w to satisfy the A_p condition, but have included the necessity for completeness.

3. Trigonometrically well-bounded operators. Given a Banach space X, a *spectral family* in X is a projection-valued function $E(\cdot) : \mathbf{R} \to B(X)$ such that

(i) $\sup\{\|E(\lambda)\| : \lambda \in \mathbf{R}\} < \infty$;
(ii) $E(\lambda)E(\mu) = E(\mu)E(\lambda) = E(\lambda)$ for $-\infty < \lambda \leq \mu < \infty$;

(iii) $E(\cdot)$ is right continuous in the strong operator topology;
(iv) $E(\lambda^-) = \lim_{\mu \to \lambda^-} E(\mu)$ exists for every $\lambda \in \mathbf{R}$;
(v) $E(\lambda) \to I$ as $\lambda \to \infty$ and $E(\lambda) \to 0$ as $\lambda \to -\infty$,

the limits in (iv) and (v) being with respect to the strong operator topology. A spectral family $E(\cdot)$ is said to be *concentrated on* $[a,b]$ if $E(\lambda) = 0$ for $\lambda < a$ and $E(\lambda) = I$ for $\lambda \geq b$.

Given a spectral family $E(\cdot)$ concentrated on $[0, 2\pi]$ and $f \in \mathrm{AC}[0, 2\pi]$, the integral $\int_{[0,2\pi]} f(\lambda)\, dE(\lambda)$ exists in the strong operator topology as a Riemann–Stieltjes integral, and the mapping $\Phi : \mathrm{AC}[0, 2\pi] \to B(X)$ defined by

$$\Phi(f) = f(0)E(0) + \int_{[0,2\pi]} f(\lambda)\, dE(\lambda) \tag{3.1}$$

is a continuous identity-preserving algebra homomorphism. The right-hand side of (3.1) is denoted by $\int_{[0,2\pi]}^{\oplus} f(\lambda)\, dE(\lambda)$. An operator V in $B(X)$ is said to be *trigonometrically well-bounded* if there is a spectral family $E(\cdot)$ concentrated on $[0, 2\pi]$ such that

$$V = \int_{[0,2\pi]}^{\oplus} e^{i\lambda}\, dE(\lambda). \tag{3.2}$$

The spectral family $E(\cdot)$ can be normalized so that $E(2\pi^-) = I$ and this normalization determines $E(\cdot)$ uniquely (for a given V). We then call $E(\cdot)$ the *spectral decomposition* of V. If V is a trigonometrically well-bounded operator with spectral decomposition $E(\cdot)$, the operator $\arg V = \int_{[0,2\pi]}^{\oplus} \lambda\, dE(\lambda)$ is called the *argument* of V. It is a well-bounded operator of type (B) [5, Definition 16.8, p. 315] and satisfies $e^{i \arg V} = V$. The existence of its argument shows, in particular, that a trigonometrically well-bounded operator is necessarily invertible. The classes of well-bounded and trigonometrically well-bounded operators provide effective generalizations to Banach spaces of, respectively, selfadjoint and unitary operators on Hilbert spaces. For a more complete account of their theory and further references, see [1] and [5].

The trigonometrical well-boundedness of an invertible operator V on a Banach space can be characterized in terms of the mapping $q \to q(V)$ defined for trigonometric polynomials q, where $q(V) = \sum \alpha_k V^k$ if $q(e^{it}) = \sum \alpha_k e^{ikt}$. This gives a rationale for the terminology (see [1, Corollary (2.17)]). In general, the characterization involves a compactness property with respect to the weak operator topology, but takes on the following simpler form when the underlying space is reflexive (see also [3, Corollary (2.3)]).

(3.3) THEOREM. *Let V be an invertible operator on a reflexive Banach space. Then V is trigonometrically well-bounded if and only if there is a constant K such that $\|q(V)\| \leq K \|q\|_{\mathbf{T}}$ for all trigonometric polynomials q.*

We shall also need a characterization of trigonometrical well-boundedness (again for operators on reflexive spaces) which indicates a connection with the discrete Hilbert transform. To state this precisely, let

$$s_n(V,t) = \sum_{k=-n}^{n}{}' k^{-1} e^{ikt} V^k$$

for an invertible operator V, $t \in [0, 2\pi]$ and $n \in \mathbf{N}$. Notice that $s_n(V,t)$ can be viewed as the operator obtained by transferring in the sense of Coifman and Weiss [4] the kernel on $\mathbf{Z}$ associated with the nth truncate of the discrete Hilbert transform by the representation $k \to e^{-ikt}V^{-k}$ of $\mathbf{Z}$ in X.

(3.4) THEOREM ([2, Corollary (2.10)]). *Let V be an invertible operator on a reflexive Banach space. If*

$$\sup\{\|s_n(V,t)\| : n \in \mathbf{N} \text{ and } t \in [0,2\pi]\} < \infty,$$

then V is trigonometrically well-bounded.

4. Weighted shifts.

We are now in a position to discuss the main result of this note, namely to give necessary and sufficient conditions under which a bilateral weighted shift on a reflexive l^p space is trigonometrically well-bounded. Before stating the result in detail, some remarks concerning the term "weighted shift" are in order, since it is used in two distinct, albeit similar, ways.

For $1 \le p < \infty$, let l^p denote the standard two-sided unweighted sequence space of all p-summable complex sequences. A *weighted shift* on l^p is an operator S_α of the form

$$S_\alpha(\{x_k\}) = \{\alpha_k x_{k-1}\}, \tag{4.1}$$

where $\alpha = \{\alpha_k\}_{k\in\mathbf{Z}}$ is a sequence of scalars. The condition that (4.1) does indeed define a bounded linear mapping of l^p into itself is that the sequence α is bounded, and S_α is an invertible element of $B(l^p)$ if and only if α is both bounded and bounded away from 0. Further, S_α is isometrically similar to S_β, where $\beta_k = |\alpha_k|$ for $k \in \mathbf{Z}$, and so in many situations only weighted shifts with non-negative weights need be considered.

When the weight sequence α has no term equal to zero, an alternative way to analyze S_α is to note that it is isometrically similar to the unweighted shift U defined on an appropriate weighted sequence space $l^p(w)$. More precisely, let $\alpha = \{\alpha_k\}_{k\in\mathbf{Z}}$ be a bounded sequence of non-zero scalars, define γ_k for $k \in \mathbf{Z}$ by

$$\gamma_k = \begin{cases} (\alpha_1 \dots \alpha_k)^{-1} & \text{if } k > 0, \\ 1 & \text{if } k = 0, \\ (\alpha_{k+1} \dots \alpha_0) & \text{if } k < 0, \end{cases}$$

and let $w_k = |\gamma_k|^{-p}$. Then the mapping T defined by $T\{x_k\} = \{\gamma_k x_k\}$ is an isometric isomorphism of l^p onto $l^p(w)$ satisfying $UT = TS_\alpha$. In the converse direction, it is easy to show that the unweighted bilateral shift on a space $l^p(w)$ is isometrically similar to an appropriate weighted shift S_α on l^p. In the context of the present paper, it is more convenient to consider a weighted shift as the operator U on $l^p(w)$ for some weight w.

Following these preliminary remarks, we now state the main result of the paper, giving necessary and sufficient conditions for the bilateral shift U on a reflexive $l^p(w)$ space to be trigonometrically well-bounded.

(4.2) THEOREM. *Let $1 < p < \infty$, let $w = \{w_k\}_{k\in\mathbf{Z}}$ be a sequence of positive weights, and let U denote the bilateral shift $U\{x_k\} = \{x_{k-1}\}$ on $l^p(w)$. Then the following statements are equivalent.*

(i) *w satisfies the A_p condition.*
(ii) *U is trigonometrically well-bounded.*
(iii) *U is bounded and invertible, and*

$$\sup\{\|s_n(U,0)\| : n \in \mathbf{N}\} < \infty.$$

(iv) *U is bounded and invertible, and $s_n(U,0)$ converges in the strong operator topology as $n \to \infty$.*

Proof. (i) $\Rightarrow$ (iii). Suppose that w satisfies the A_p condition. Taking $I = \{k, k+1\}$ in (2.1), it is seen that

$$w_k/w_{k+1} \leq (w_k + w_{k+1})(w_k^{-1/(p-1)} + w_{k+1}^{-1/(p-1)})^{p-1} \leq 2^p C_p$$

for all $k \in \mathbf{Z}$. Similarly, $w_{k+1}/w_k \leq 2^p C_p$ for all $k \in \mathbf{Z}$. Therefore, by (1.2), U is bounded and invertible. Since $s_n(U,0)x = h_n * x$ for x in $l^p(w)$, condition (iii) in Theorem (2.3) gives the existence of a constant K_p such that $\|s_n(U,0)\| \leq K_p$ for all $n \in \mathbf{N}$. This establishes (iii).

(iii) $\Rightarrow$ (ii) and (iv). Suppose that

$$M \equiv \sup\{\|s_n(U,0)\| : n \in \mathbf{N}\} < \infty.$$

For $t \in [0, 2\pi]$, the mapping $W_t : l^p(w) \to l^p(w)$ defined by $W_t\{x_k\} = \{e^{ikt}x_k\}$ is an invertible isometry and $e^{it}U = W_t U W_t^{-1}$. It follows that

$$M = \sup\{\|s_n(U,t)\| : n \in \mathbf{N}\} < \infty$$

for $t \in [0, 2\pi]$ and therefore, by Theorem (3.4), U is trigonometrically well-bounded. Now [2, Corollary (3.9)] gives (iv).

(iv) $\Rightarrow$ (iii). This is clear from the principle of uniform boundedness.

(ii) $\Rightarrow$ (i). Suppose that U is trigonometrically well-bounded, let $E(\cdot)$ be the spectral decomposition of U, and let Φ be the associated continuous homomorphism of AC$[0, 2\pi]$ into $B(l^p(w))$ as in (3.1). Define $\psi : [0, 2\pi] \to \mathbf{C}$ by setting $\psi(t) = i(\pi - t)$ for $0 < t < 2\pi$, $\psi(0) = \psi(2\pi) = 0$, and let

$\{f_m\}_{m\in\mathbb{N}}$ be a sequence of trigonometric polynomials such that

$$\text{(4.3)}\qquad \begin{aligned} &\text{(a)}\quad f_m \to \psi \text{ pointwise on } [0,2\pi];\\ &\text{(b)}\quad K \equiv \sup\{\|f_m\|_{[0,2\pi]} : m \in \mathbb{N}\} < \infty. \end{aligned}$$

Let $x = \{x_k\}$ and $y = \{y_k\}$ be elements of l_0, with $x_k = y_k = 0$ for $|k| > M$, and temporarily fix $m \in \mathbb{N}$. Use $(\cdot,\cdot)$ to denote the standard pairing of $l^p(w)$ and its dual space $l^{p'}(w)$, and let $\hat{f}_m(k) = 0$ for $|k| > N > 2M$. Then

$$\text{(4.4)}\qquad \Big(\sum_{k=-N}^{N} \hat{f}_m(k)U^k x, y\Big) = \sum_{k=-N}^{N}\sum_{l=-M}^{M} \hat{f}_m(k)\, x_{l-k} y_l w_l.$$

It follows from (4.4) that

$$\text{(4.5)}\qquad \sum_{k=-2M}^{2M}\sum_{l=-M}^{M} \hat{f}_m(k)\, x_{l-k} y_l w_l = (\Phi(f_m)x, y)$$

and hence, by (4.3)(b), that

$$\text{(4.6)}\qquad \Big|\sum_{k=-2M}^{2M}\sum_{l=-M}^{M} \hat{f}_m(k)\, x_{l-k} y_l w_l\Big| \le K\|\Phi\|\,\|x\|_{w,p}\|y\|_{w,p'}.$$

By (4.3) and dominated convergence, $\hat{f}_m$ converges pointwise on $\mathbb{Z}$ to $\hat{\psi}$, the discrete Hilbert kernel h. Letting $m \to \infty$ in (4.6), we now see that

$$\text{(4.7)}\qquad |(Hx, y)| \le K\|\Phi\|\,\|x\|_{w,p}\,\|y\|_{w,p'}.$$

Since (4.7) is valid for all $y \in l_0$, it follows that $\|Hx\|_{w,p} \le K\|\Phi\|\,\|x\|_{w,p}$. This final inequality holds for all $x \in l_0$ and thus, by Theorem (2.3), w satisfies the A_p condition as required. This completes the proof of the theorem.

SCHOLIUM. *Let $1 < p < \infty$ and let $w = \{w_k\}_{k\in\mathbb{Z}}$ be a sequence of positive weights satisfying the A_p condition. Then $\sum_{k=-\infty}^{\infty} w_k = \infty$.*

Proof. If $\sum_{k=-\infty}^{\infty} w_k < \infty$, then the sequence $a = \{a_k\}_{k\in\mathbb{Z}}$ belongs to $l^p(w)$, where $a_k = 1$ for all $k \in \mathbb{Z}$. By Theorem (2.3)(ii) (and the remarks following Theorem (2.3)) applied to a, we obtain the absurd conclusion that $\sum_{m=-\infty}^{\prime\infty} |m|^{-1} < \infty$.

(4.8) COROLLARY. *Suppose that the conditions* (i)–(iv) *in Theorem* (4.2) *hold. Then the limit in the strong operator topology of $s_n(U,0)$ is the discrete Hilbert transform H on $l^p(w)$, and*

$$\arg U = \pi I + iH.$$

Proof. Since $s_n(U,0)x = h_n * x$ for $x \in l^p(w)$, the first conclusion is immediate from Theorem (2.3) and Theorem (4.2)(iv).

We now use the notation employed in the proof of the implication (4.2)(ii) $\Rightarrow$ (4.2)(i). Put $A = \arg U = \int_{[0,2\pi]}^{\oplus} \lambda\, dE(\lambda)$. If $x \in l^p(w)$ and $E(0)x = x$, then clearly $Ax = 0$, and consequently $Ux = e^{iA}x = x$. It follows that x is a constant sequence and so, by the preceding Scholium, $x = 0$. We have shown that $E(0) = 0$, whence it is readily inferred, with the aid of (4.3)(a), (b) and [5, Theorem 17.5, p. 337], that

$$\lim_m \Phi(f_m) = i\pi I - iA,$$

the limit here being in the strong operator topology. Letting $m \to \infty$ in (4.5) identifies this strong limit as H, and completes the proof.

The above proof that (i) implies (ii) is not direct, but relies on the results of [2] and the maximal inequality of Theorem (2.3)(iii) associated with the A_p condition. A more economical approach is in fact possible, using the earlier and more direct characterization of trigonometrical well-boundedness given by Theorem (3.3), together with the following variant of the well known result of Stechkin concerning the boundedness on l^p of convolution by the sequence of Fourier coefficients of a function in $\mathrm{BV}(\mathbf{T})$ when $1 < p < \infty$ (see [5, Theorem 20.7, pp. 377–378]).

(4.9) THEOREM. *Let $1 < p < \infty$, let the weight sequence w satisfy the A_p condition, and let $\phi \in \mathrm{BV}(\mathbf{T})$. Then the convolution operator $x \to \hat{\phi} * x$ is bounded on $l^p(w)$ and has norm bounded by $K_{p,w}\|\phi\|_{\mathbf{T}}$, where $K_{p,w}$ is a constant depending only on the norm of the discrete Hilbert transform H acting on $l^p(w)$.*

Proof. This is a simple adaptation of the original proof of Stechkin's result as presented in [5].

It follows immediately from Theorem (4.9) that, if w satisfies the A_p condition with p in the range $1 < p < \infty$, then $\|q(U)\| \leq K_{p,w}\|q\|_{\mathbf{T}}$ for all trigonometric polynomials q, when U is considered as acting on $l^p(w)$. Hence, by Theorem (3.3), U is trigonometrically well-bounded.

5. The norm growth of iterates and trigonometrical well-boundedness. A well known result of B. Sz.-Nagy [5, Theorem 8.1, p. 188] implies that an invertible operator V on a Hilbert space H is similar to a unitary operator if and only if $\|V^n\| = O(1)$ as $|n| \to \infty$, whilst a related result of J. Wermer asserts that a bounded operator on H is a scalar-type spectral operator if and only if it is similar to a normal operator [5, Theorem 8.3, p. 190]. Putting these results together, it is seen that V is a scalar-type spectral operator with spectrum contained in $\mathbf{T}$ if and only if $\|V^n\| = O(1)$ as $|n| \to \infty$.

Since trigonometrically well-bounded operators are invertible and have a spectral diagonalization (3.2) reminiscent of (but in general weaker than) that of scalar-type spectral operators, it is natural to ask whether trigonometric well-boundedness for an invertible operator V on a Hilbert space can likewise be characterized by the behaviour of $\|V^n\|$. One simple necessary condition on the growth rate of $\|V^n\|$ for trigonometrical well-boundedness is that $\|V^n\| = O(|n|)$ as $|n| \to \infty$. This is easily seen by noting that, in the notation of (3.1), $V^n = \Phi(\chi_n)$, where $\chi_n(t) = e^{int}$, and $\|\chi_n\|_{\mathbf{T}} = 2\pi|n| + 1$. However, as Example (5.2) below illustrates, this growth rate on $\|V^n\|$ is not sufficient for trigonometrical well-boundedness. Indeed, further examples are given to show that no growth rate on $\|V^n\|$ can simultaneously give a necessary and sufficient condition for trigonometrical well-boundedness.

(5.1) EXAMPLE. Define the weight sequence $w^{(1)}$ by setting $w_k^{(1)} = |k|^\alpha$ for $k \in \mathbf{Z} \setminus \{0\}$ and $w_0^{(1)} = 1$, where $0 < \alpha < 1$. For $1 \le s < t < \infty$,

$$\Big(\int_s^t x^\alpha\,dx\Big)\Big(\int_s^t x^{-\alpha}\,dx\Big) = \frac{s^2 + t^2 - s^{1-\alpha}t^{1+\alpha} - t^{1-\alpha}s^{1+\alpha}}{1-\alpha^2} \le \frac{(t-s)^2}{1-\alpha^2},$$

since $s^{1-\alpha}t^{1+\alpha} + t^{1-\alpha}s^{1+\alpha} \ge 2st$. Therefore, by (2.4)(iv), $w^{(1)}$ satisfies the A_2 condition on $\mathbf{N}$, and hence on $\mathbf{Z}$ by (2.4)(ii). Let U_1 denote the bilateral shift on $l^2(w^{(1)})$. Then U_1 is trigonometrically well-bounded and $\|U_1^n\| = (1+|n|)^{\alpha/2}$ for $n \in \mathbf{Z}$ by (1.3).

(5.2) EXAMPLE. Define $w^{(2)}$ by setting $w_k^{(2)} = k^\alpha$ for $k \ge 1$ and $w_k^{(2)} = 1$ for $k \le 0$, where $0 < \alpha < 1$. Then the bilateral shift U_2 on $l^2(w^{(2)})$ is bounded and invertible by (1.2), and satisfies

$$\|U_2^n\| = (1+n)^{\alpha/2} \quad (n \ge 0), \qquad \|U_2^n\| = 1 \quad (n < 0)$$

by (1.3). Thus $\|U_2^n\| = O(|n|)$ as $|n| \to \infty$. However, $w^{(2)}$ does not satisfy the A_2 condition by (2.4)(iii) and so U_2 is not trigonometrically well-bounded.

(5.3) EXAMPLE. Let $V = U_2 \oplus U_2^{-1}$ on $l^2(w^{(2)}) \oplus l^2(w^{(2)})$. Then V is bounded and invertible, and $\|V^n\| = (1+|n|)^{\alpha/2}$ for all $n \in \mathbf{Z}$. However, V is not trigonometrically well-bounded. This is easily seen by noting that, if it were, then its restriction U_2 to the first direct summand would also be trigonometrically well-bounded by Theorem (3.3), contradicting (5.2).

Examples (5.1) and (5.3) thus give two invertible Hilbert space operators U_1 and V, the former trigonometrically well-bounded and the latter not, with $\|U_1^n\| = \|V^n\|$ for all $n \in \mathbf{Z}$. Hence there can be no condition on the norms of the iterates of a Hilbert space operator which is both necessary and sufficient for trigonometrical well-boundedness.

(5.4) Remark. It was shown in [2, Theorem (1.2)] that, for a trigonometrically well-bounded operator V on a reflexive space, the norm boundedness of $\{V^n : n \in \mathbb{Z}\}$ implies that

$$\sup\{\|s_n(V,t)\| : n \in \mathbb{N},\ 0 \le t \le 2\pi\} < \infty.$$

Example (3.13) in [2] showed that the converse implication was not valid. That example was on a non-Hilbert reflexive space. The above Example (5.1), taken with Theorem (4.2) and the fact that $\|s_n(U,0)\| = \|s_n(U,t)\|$ for $n \in \mathbb{N}$ and $0 \le t \le 2\pi$ when U is a bilateral shift (see the proof of the implication (iii) $\Rightarrow$ (ii) and (iv) in Theorem (4.2)), provides an example on Hilbert space of the failure of this converse implication.

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