COLLOQUIUM MATHEMATICUM

VOL. XV 1966 FASC. 2

ON SOME TERNARY RELATIONS IN LATTICES

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A *betweenness relation* in a set is a ternary relation defined in the set satisfying certain postulates. G. Birkhoff in one of his problems gives such a set of conditions (Problem 1 in [3]). In lattices several notions of betweenness are known (see e.g. [2]). But it seems that the interrelations among these ternary relations have not been studied so far. Moreover, some of these ternary relations do not, in general, satisfy certain known conditions of betweenness. In this note (1) we consider the interrelations among these ternary relations and discuss their connection with the Birkhoff conditions of betweenness. These discussions lead us to some interesting characterizations of modular, distributive and chain lattices. As a consequence we obtain some of the results proved by Ellis and Blumenthal [2].

DEFINITION.

$A(axb)$ means $a \leqslant x \leqslant b$ or $b \leqslant x \leqslant a$,

$B(axb)$ means $ax+xb = x = (a+x)(x+b)$,

$C(axb)$ means $ax+xb = x = ab+x$,

$C^*(axb)$ means $(a+x)(x+b) = x = (a+b)x$,

$D(axb)$ means $ab \leqslant x \leqslant a+b$,

where $a, x, b \in L$, $(L, +, \cdot, \leqslant)$ being a lattice.

The relation B was given by Glivenko as the lattice-theoretic characterization of metric betweenness, and Pitcher and Smiley [4] adapted this as the definition of betweenness in arbitrary lattices. The relations C and C^* which are non-self dual characterizations of metric betweenness are due to Blumenthal and Ellis [3] (2).

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(1) Presented to the 30th conference of the Indian Mathematical Society.

(2) It is to be noted that the relations B, C, C^* and D are the only possible independent combinations of two or more of the equations $ax+xb = x$, $(a+x)(x+b) = x$, $ab+x = x$ and $(a+b)x = x$.

THEOREM 1. *In any lattice the following implication scheme holds:*

$$A \Rightarrow B \begin{smallmatrix} \nearrow C \searrow \\ \searrow C^* \nearrow \end{smallmatrix} D.$$

By $A \Rightarrow B$ *we mean "if* $A(axb)$, *then* $B(axb)$".

The proof being simple is omitted.

We will refer the following propositional functions 1 - 5 as the *Birkhoff conditions of betweenness*:

1. $T(axb) \Rightarrow T(bxa)$.
2. $T(axb),\ T(abx) \Rightarrow x = b$.
3. $T(ayx),\ T(axb) \Rightarrow T(ayb)$.
4. $T(axb),\ T(xby),\ x \neq b \Rightarrow T(aby)$.
5. $T(abc),\ T(acd) \Rightarrow T(bcd)$.
6. $T(abc),\ T(adc),\ T(bed) \Rightarrow T(aec)$.

The sixth condition will be referred to as *Blumenthal condition.*

THEOREM 2. *In a lattice* L *the following statements are equivalent:*

1. L *is a chain.*
2. $D \Rightarrow A$.
3. B *satisfies the fourth Birkhoff condition.*

Proof. 1 $\Rightarrow$ 2. Since in a chain we have either $ab = a,\ a+b = b$ or $ab = b, a+b = a$, it follows that if $D(axb)$, then $A(axb)$.

2 $\Rightarrow$ 3. If $D \Rightarrow A$, then, by Theorem 1, A and B are equivalent. A is easily seen to satisfy the fourth Birkhoff condition and so B satisfies the same.

3 $\Rightarrow$ 1. Let B satisfy the fourth condition. Let a, b be two distinct elements of L. If $a+b = b$, then $a \leqslant b$. Let $a+b \neq b$. Now put $a+b = x$; $ab = y$ and note that $B(axb)$ and $B(xby)$ and $x \neq y$ are true. It follows from the fourth condition that $ab+bab = b$, i.e. $ab = b$, which shows that L is a chain.

COROLLARY. *A lattice* L *is a chain if and only if the conditions* A, B, C, C^*, *and* D *are equivalent in* L.

THEOREM 3. *In a lattice* L *the following statements are equivalent:*

1. L *is a distributive lattice.*
2. B *satisfies the Blumenthal condition.*
3. $D \Rightarrow B$.
4. D *satisfies the second Birkhoff condition.*

Proof. 1 $\Rightarrow$ 2. Let L be a distributive lattice and let $B(abc)$, $B(adc)$ and $B(bed)$. Since, by Theorem 1, $B \Rightarrow D$, we have $D(abc)$, $D(adc)$ and $D(bed)$. This implies $ac \leqslant e \leqslant a+c$ and hence, since the lattice is distributive, also $B(aec)$.

$2 \Rightarrow 3$. Let B satisfy the Blumenthal condition and let $D(axb)$. Since order betweenness implies B, we have $B(ab, x, a+b)$. Also we have $B(a, ab, b)$ and $B(a, a+b, b)$. Hence by the assumed condition we have $B(axb)$.

$3 \Rightarrow 4$. If $D \Rightarrow B$, D and B are equivalent by Theorem 1, and it is easily seen that B and hence D satisfies the second Birkhoff condition.

$4 \Rightarrow 1$. Let D satisfy the second condition. Let $ax = ab$, $a+x = a+b$. So we have $D(axb)$ and $D(abx)$. Hence $x = b$. Thus, by a known criterion of distributivity, the lattice is distributive.

COROLLARY. *A lattice L is distributive if and only if the relations B, C, C^* and D are equivalent in L.*

THEOREM 4. *In a lattice L the following statements are equivalent:*

1. *L is modular.*
2. *B satisfies the third Birkhoff condition.*
3. *$C \Rightarrow B$.*
4. *$C \Rightarrow C^*$.*

Proof. $1 \Rightarrow 2$. Let the lattice be modular and let $B(ayx)$ and $B(axb)$. We will show that $B(ayb)$.

Since $B(ayx)$ we have	Similarly, $B(axb)$ will mean
(i) $ay+yx = y$,	(v) $ax+xb = x$,
(ii) $(a+y)(y+x) = y$,	(vi) $(a+x)(x+b) = x$,
(iii) $ax+y = y$,	(vii) $ab+x = x$,
(iv) $(a+x)y = y$,	(viii) $(a+b)x = x$.

Now put $Y = ay+yb$ and $Y' = (a+y)(y+b)$. To show $B(ayb)$ we need only show that $Y = y = Y'$. It is clear that $Y \leqslant y \leqslant Y'$. Now, $aY = a(yb+ay) = ayb+ay$ using the modular law, and hence $aY = ay$. Also,

$$\begin{aligned}
a+Y &= a+ay+yb \\
&= a+yb \\
&= a+b(a+y)(y+x) \text{ (by (ii))} \\
&= \big(a+b(y+x)\big)(a+y) \text{ (by modular law)} \\
&= \Big(a+b\big(y+(x+b)(a+x)\big)\Big)(a+y) \text{ (by (vi))} \\
&= \big(a+b(y+x+b)(a+x)\big)(a+y) \text{ (by (iv) and modular law)} \\
&= \big(a+b(a+x)\big)(a+y) \\
&= (a+b)(a+x)(a+y) \text{ (by modular law)} \\
&= \big((a+b)a+x\big)(a+y) \text{ (by (viii) and modular law)} \\
&= (a+x)(a+y) \\
&= (a+y) \text{ (by (iv) and modular law).}
\end{aligned}$$

Thus we have $Y \leqslant y$, $aY = ay$ and $a+Y = a+y$. Since the lattice is modular it follows that $Y = y$ and similarly by the dual argument we can prove that $Y' = y$. So we have $B(ayb)$.

$2 \Rightarrow 1$. Let B satisfy the third condition and let $ax = ab$, $a+x = a+b$, $x \leqslant b$. We need only show that $x = b$. We have $x \leqslant b \leqslant a+x$ and hence $B(x, b, a+x)$. Since $B(x, a+x, a)$ is always true, we have, by the assumed condition, $B(xba)$ which implies $xb+ba = b$ and hence $x = b$.

It is clear from definition and Theorem 1 that 3 and 4 are equivalent. By a result of Blumenthal ([2], p. 318) we know that $1 \Rightarrow 3$.

We will now prove that $3 \Rightarrow 1$. Let again $ax = ab$, $a+x = a+b$, $x \leqslant b$. We then have $C(axb)$ and hence we get $B(axb)$ which implies that $(a+x)(x+b) = x$, i.e. $x = b$. So the lattice is modular. The proof of the theorem is complete.

As mentioned earlier, it was Blumenthal and Ellis who first observed that in a modular lattice the relations B, C and C^* are equivalent. Here we get the converse, too.

In conclusion, I would like to express my sincere thanks to Professor M. Venkataraman under whose guidance this note was prepared.

REFERENCES

[1] G. Birkhoff, *Lattice theory* (2nd ed.), American Mathematical Society Colloquium Publications 25 (1961).

[2] M. Blumenthal, *Distance geometry*, Oxford 1953.

[3] — and D. O. Ellis, *Notes on lattices*, Duke Mathematical Journal 16 (1949), p. 585-590.

[4] E. Pitcher and F. Smiley, *Transitivities of betweenness*, Transactions of the American Mathematical Society 52 (1942), p 95-114.

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Reçu par la Rédaction le 4. 2. 1965