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ON SATURATING ULTRAFILTERS ON N

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The main result of this paper is essentially a theorem about ultraproducts of countable sets. It is known that if the Continuum Hypothesis is assumed, then ultraproducts of countable structures are saturated for all free ultrafilters, and Ellentuck and Rucker [4] proved that if Martin's Axiom is assumed, then there exist $2^{\mathfrak{c}}$ such ultrafilters.

In this paper we regard only ultraproducts of countable sets. We prove that if $P(\mathfrak{c})$ is assumed, then there exist $2^{\mathfrak{c}}$ saturating ultrafilters on N which are absolute and minimal points in N^*.

1. Preliminaries. Let N denote the set of all positive integers, and $N^* = \beta N \setminus N$ the set of all free ultrafilters on N. Recall that an ultrafilter $\mathscr{F} \in N^*$ is a *P-point* if each map $f\colon N \to N$ is either constant or finite-to-one on some element of $\mathscr{F}$. The ultrafilter $\mathscr{F} \in N^*$ is *minimal* with respect to Rudin-Keisler ordering if each map $f\colon N \to N$ is either constant or one-to-one on some element of $\mathscr{F}$. Hence every minimal point is a P-point in N^*.

Assume that A and B are subsets of N. We say that A is *almost contained* in B, $A < B$, if $A \setminus B$ is finite.

A point $\mathscr{F}$ of N^* is said to be an *absolute ultrafilter* if it has a base linearly ordered with respect to $<$.

We shall use the following consequence of Martin's Axiom which will be called $P(\mathfrak{c})$ (see, e.g., Booth [1]).

$P(\mathfrak{c})$: *If $\mathscr{F}$ is a base of a free filter on N and* $\operatorname{card}\mathscr{F} < \mathfrak{c}$, *then there exists an infinite subset T of N such that $T < A$ for every A from $\mathscr{F}$.*

Recall that $P(\mathfrak{c})$ implies $2^{\aleph} = 2^{\aleph_0}$ for each cardinal $\aleph$ such that $\aleph_0 \leqslant \aleph < 2^{\aleph_0}$ (Booth [2]).

Assertion $P(\mathfrak{c})$ is claimed by Kunen and Tall [5] to be essentially weaker than Martin's Axiom.

Let X be the product of sets X_n ($n \in N$), and let $\mathscr{F}$ be a filter on N. We define a relation $=_{\mathscr{F}}$ on X as follows:

$$a =_{\mathscr{F}} b \text{ if } \{n \in N\colon a_n = b_n\} \in \mathscr{F}.$$

The *ultraproduct of sets* X_n $(n \in N)$ with respect to the filter $\mathscr{F}$ is the set $X/\mathscr{F}$, i.e. the set of all equivalence classes of $=_{\mathscr{F}}$.

Let $\leqslant$ be a relation on X induced by the relations $\leqslant_n$ on X_n as follows:

$$a \leqslant b \quad \text{if } a_n \leqslant_n b_n \text{ for each } n \in N.$$

Let the same symbol $\leqslant$ denote the relation induced by $\leqslant_n$ $(n \in N)$ on $X/\mathscr{F}$:

$$[a]_{\mathscr{F}} \leqslant [b]_{\mathscr{F}} \quad \text{if } \{n \in N: a_n \leqslant_n b_n\} \in \mathscr{F}.$$

A filter $\mathscr{F}$ on N is said to be $\mathfrak{c}$-*saturating* if the following condition is satisfied:

If A and B are subsets of $X/\mathscr{F}$ of cardinality less than $\mathfrak{c}$, $\leqslant$ is a relation induced by $\leqslant_n$ $(n \in N)$, and for any finite subsets $A' \subset A$ and $B' \subset B$ there exists a $c' \in X/\mathscr{F}$ such that $a' \leqslant c' \leqslant b'$ for $a' \in A'$ and $b' \in B'$, then there exists a $c \in X/\mathscr{F}$ such that $a \leqslant c \leqslant b$ for $a \in A$ and $b \in B$.

If we take a partition $\{X_n: n \in N\}$ of N and a free ultrafilter $\mathscr{F}$ on N, then the ultraproduct $X/\mathscr{F}$ is a subset of N^*. Also, ultraproducts of rational numbers can be regarded as subsets of their remainder of the Čech-Stone compactification of the non-negative part of the real line. Recently, Mioduszewski [6] proved that the existence of saturating and absolute ultrafilters implies that there exist remote and absolute points in R^*.

Ultraproducts are also used for other purposes and in more general setting in the theory of models (see, e.g., Chang and Keisler [3]).

2. Basic Lemma. Let X be a product of the sets X_n $(n \in N)$, and let $\leqslant$ be a relation on X induced by the relations $\leqslant_n$ on X_n. For $F \subset N$ we write $a \leqslant_F b$ if $a_n \leqslant_n b_n$ for all but finitely many n from F. Let $\mathscr{F}$ be a family of subsets of N. We write $a \leqslant_{\mathscr{F}} b$ if $a \leqslant_F b$ for some $F \in \mathscr{F}$. The relation $\leqslant_{\mathscr{F}}$ on X induces the relation $\leqslant$ in the ultraproduct $X/\mathscr{F}$ whenever $\mathscr{F}$ is a filter; i.e., $a \leqslant_{\mathscr{F}} b$ implies $[a]_{\mathscr{F}} \leqslant [b]_{\mathscr{F}}$. The relation $\leqslant$ will be said to be *dense* with respect to subsets A and B of X on a family $\mathscr{F}$ if the following condition is satisfied:

(*) For arbitrary finite subsets $A' \subset A$ and $B' \subset B$ there exists a $c \in X$ such that $a \leqslant_{\mathscr{F}} c \leqslant_{\mathscr{F}} b$ for $a \in A'$ and $b \in B'$.

If $\mathscr{F} = \{F\}$ is a one-element family, then we say that $\leqslant$ is *dense* with respect to A and B on F.

LEMMA. *Assume $P(\mathfrak{c})$. Let X be the product of countable sets X_n $(n \in N)$, and let $\leqslant$ be a relation on X induced by relations $\leqslant_n$ on X_n. Let A and B be subsets of cardinality less than $\mathfrak{c}$ such that the relation $\leqslant$ is dense with respect to A and B on N. Then there exist an infinite subset T of N and $c \in X$ such that $a \leqslant_T c \leqslant_T b$ for each $a \in A$ and $b \in B$.*

Proof. Let $X' = \bigcup \{\{n\}\times X_n\colon n\in N\}$, and for $a\in X$ let

$$R(a) = \bigcup_{n\in N} \{(n, c_n)\colon a_n \leqslant_n c_n\} \quad \text{and} \quad L(a) = \bigcup_{n\in N} \{(n, c_n)\colon c_n \leqslant_n a_n\}.$$

For $M\subset N$, let $X'_M = \bigcup \{\{n\}\times X_n\colon n\in M\}$. The family $\mathscr{G}$ consisting of all sets $R(a)$ and $L(b)$ for $a\in A$ and $b\in B$ and of all sets $X'_{N\setminus C}$, where C are finite subsets of N, is centered, i.e., it is a subbase of a free filter on X'. In fact, let $A'\subset A$ and $B'\subset B$ be finite and let $C_1, \ldots, C_m$ be finite subsets of N. There exists an element $c\in X$ such that

$$a\leqslant_N c\leqslant_N b \quad \text{for any } a\in A' \text{ and } b\in B',$$

i.e., there exists a finite subset C of N such that

$$a_n\leqslant_n c_n\leqslant_n b_n \quad \text{for } n\in N\setminus C,\ a\in A' \text{ and } b\in B'.$$

Hence, the infinite set

$$\bigcup \Big\{(n, c_n)\colon n\in N\setminus \Big(\bigcup_{i=1}^{m} C_i\cup C\Big)\Big\}$$

is contained in the intersection

$$\bigcap \{R(a)\colon a\in A'\}\cap\bigcap \{L(b)\colon b\in B'\}\cap\bigcap \{X'_{N\setminus C_i}\colon i = 1, \ldots, m\}.$$

The set X' is countable and $\operatorname{card}\mathscr{G}\leqslant \max(\operatorname{card}A, \operatorname{card}B, \aleph_0) < \mathfrak{c}$. Hence, it follows from $P(\mathfrak{c})$ that there exists an infinite subset H of X' such that $H < G$ for each $G\in\mathscr{G}$.

Since $H < X'_{N\setminus C}$ for any finite subset C of N, the set

$$T = \{n\in N\colon H\cap X'_n \neq \emptyset\}$$

is infinite.

Choose a $c\in X$ such that $c_n\in H\cap X'_n$ if $n\in T$.

The set T and the element c in X are as required. In fact, if $a\in A$, then $H < R(a)$, and so there exists a finite subset C of N such that $H\setminus X'_C \subset R(a)$, i.e., $a_n\leqslant_n c_n$ for $n\in T\setminus C$, with finite C. This means that $a\leqslant_T c$ for $a\in A$. Similarly, $c\leqslant_T b$ for $b\in B$.

Note. The Lemma can be easily relativized to the case of arbitrary infinite F, $F\subset N$. Namely, if $\leqslant$ is dense with respect to A and B on F, then the set T from the conclusion of the Lemma is a subset of F.

3. THEOREM. *Assume $P(\mathfrak{c})$. Let X be a product of countable sets X_n $(n\in N)$, and let $\mathscr{R}$ be the family of all relations on X which are induced by relations on coordinate sets. Then there exist $2^{\mathfrak{c}}$ absolute and minimal ultrafilters $\mathscr{F}$ on N such that if A and B are subsets of X of cardinality less than $\mathfrak{c}$, and a relation $\leqslant$ from $\mathscr{R}$ is dense with respect to A and B on $\mathscr{F}$, then there exist $d\in X$ and $F\in\mathscr{F}$ such that $a\leqslant_F d\leqslant_F b$ for each $a\in A$ and $b\in B$.*

Proof. First we construct only one such ultrafilter $\mathscr{F}$.

Denote by $\{U_a: a < \mathfrak{c}\}$ the family of infinite subsets of N. Let $\{f_a: a < \mathfrak{c}\}$ be the family of maps from N into itself. We have card$\mathscr{R} \leqslant \mathfrak{c}$, since there exist at most $\mathfrak{c}$ relations on X_n, X_n being countable. Take a set $\mathscr{A}$ of all triples $(A, B, \leqslant)$, where card$A < \mathfrak{c}$, card$B < \mathfrak{c}$, A, B are subsets of X, and $\leqslant$ is a relation from $\mathscr{R}$. The family of all subsets of X of cardinality less than $\mathfrak{c}$ is of cardinality $\mathfrak{c}$. This follows from the fact that $2^{\aleph} = \mathfrak{c}$ for $\aleph_0 \leqslant \aleph < \mathfrak{c}$, $P(\mathfrak{c})$ being assumed. Thus there are $\mathfrak{c}$ triples $(A, B, \leqslant)$. Let $\{(A, B, \leqslant)_a: a < \mathfrak{c}\}$ be a well ordering of the set $\mathscr{A}$ in which each triple appears $\mathfrak{c}$ times.

We construct, by transfinite induction, a family $\{F_a: a < \mathfrak{c}\}$ of infinite subsets of N satisfying the following conditions:

(1) $F_\beta < F_a$ whenever $a < \beta$.

(2) If the a-th triple $(A, B, \leqslant)_a$ is such that $\leqslant$ is dense with respect to A and B on F_a, then there exists a $d \in X$ such that

$$a \leqslant_{F_{a+1}} d \leqslant_{F_{a+1}} b \quad \text{for each } a \in A \text{ and } b \in B.$$

(3) $f_a | F_{a+1}$ is constant or one-to-one.

(4) If the intersection $F_{a+1} \cap U_a$ is infinite, then $F_{a+1} \subset U_a$.

Let $F_0 = N$. Assume that F_β's with properties (1)-(4) are already defined for $\beta < a$, $a < \mathfrak{c}$.

If a is a limit ordinal, then it follows from $P(\mathfrak{c})$ and (1) that there exist infinite subsets of N which are almost contained in F_β for $\beta < a$. Let F_a be one of them.

If $a = \beta + 1$, consider two cases:

(a) $(A, B, \leqslant)_\beta$ does not satisfy the hypothesis of condition (2) for $a = \beta$. Take any infinite subset $T \subset F_\beta$ such that $f_\beta | T$ is constant or one-to-one. Let $F_{\beta+1} = T \cap U_\beta$ if this intersection is infinite and let $F_{\beta+1} = T$ in the other case. Conditions (1)-(4) are obviously satisfied, condition (2) vacuously.

(b) $(A, B, \leqslant)_\beta$ satisfies the hypothesis of condition (2) for $a = \beta$. Applying the Lemma to $N = F_\beta$ and to A, B, and $\leqslant$ (see the Note), we obtain a point $d \in X$ and an infinite subset T of F_β such that

$$a \leqslant_T d \leqslant_T b \quad \text{for each } a \in A \text{ and } b \in B.$$

Now we construct the set $F_{\beta+1}$ by means of that T as in case (a) Thus the family $\{F_a: a < \mathfrak{c}\}$ is constructed.

Now let $\mathscr{F}$ be a filter generated by the family $\{F_a: a < \mathfrak{c}\}$. In view of conditions (1) and (4), $\mathscr{F}$ is a free ultrafilter. It follows from (1) that $\mathscr{F}$ is absolute, and from (3) that $\mathscr{F}$ is minimal.

Now let A and B be subsets of X of cardinality less than $\mathfrak{c}$ and let $\leqslant$ be a relation from $\mathscr{R}$ which is dense with respect to A and B on $\mathscr{F}$. Since $\mathscr{F}$ is absolute and $\operatorname{card} A < \mathfrak{c}$, $\operatorname{card} B < \mathfrak{c}$, there exists an α, $\alpha < \mathfrak{c}$, such that $\leqslant$ is dense with respect to A and B on a set $F_\alpha \in \mathscr{F}$. And since $(A, B, \leqslant)$ appears $\mathfrak{c}$ times in the enumeration of $\mathscr{A}$, there exists a $\beta > \alpha$ such that the β-th element of this enumeration is $(A, B, \leqslant)$. A set F_β is almost contained in F_α. Hence $\leqslant$ is dense on F_β, and it follows from (2) that there exists a $d \in X$ such that

$$a \leqslant_{F_{\beta+1}} d \leqslant_{F_{\beta+1}} b \quad \text{for each } a \in A \text{ and } b \in B.$$

To obtain $2^{\mathfrak{c}}$ suc ultrafilters we proceed in the standard way. The family of all functions $f\colon \mathfrak{c} \to \{0, 1\}$ has cardinality $2^{\mathfrak{c}}$. In the above proof, in every step, we may decompose the sets F_α into two disjoint infinite sets $F_\alpha^0 \cup F_\alpha^1$ and put $\mathscr{F}_f = \{F_\alpha^{f(\alpha)}\colon \alpha < \mathfrak{c}\}$.

Passing to ultraproducts we get the following

COROLLARY. *Assume $P(\mathfrak{c})$. If X is a product of countable sets X_n $(n \in N)$, then there exist $2^{\mathfrak{c}}$ absolute and minimal ultrafilters on N which are $\mathfrak{c}$-saturating.*

REFERENCES

[1] D. Booth, *Ult filters on a countable set*, Annals of Mathematical Logic 2 (1970-1971), p. 1-24.

[2] — *A Boolean aiew of sequential compactness*, Fundamenta Mathematicae 85 (1970), p. 99-102.

[3] C. C. Chang and H. J. Keisler, *Model theory*, North Holland 1973.

[4] E. Ellentuck and R. Rucker, *Martin's Axiom and saturated models*, Proceedings of the American Mathematical Society 34 (1972), p. 243-249.

[5] K. Kunen and F. D. Tall, *Between Martin's Axiom and Suslin's Hypothesis*, preprint.

[6] J. Mioduszewski, *On composants of $\beta R \setminus R$*, preprint.

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