COLLOQUIUM MATHEMATICUM

VOL. LX/LXI DÉDIÉ À M. ANTONI ZYGMUND 1990

A REMARK ON GRADIENTS OF HARMONIC FUNCTIONS IN DIMENSION ≥ 3

BY

J. BOURGAIN (BURES-SUR-YVETTE)
AND T. WOLFF (PASADENA, CALIFORNIA)

Introduction. The purpose of this note is to refine the result of [2] on the existence of $C^{1+\epsilon}$-harmonic functions on $\mathbf{R}_+^d$ $(d \geq 3)$ for which ∇f vanishes on a boundary set of positive measure. Using some new ingredients originating from [1], one actually gets vanishing of f, ∇f simultaneously (see the Theorem below). The problem whether this may happen for functions of class C^2 or C^∞ remains open (see the remarks at the end). The present construction is based on the same techniques as in [2] but in this case is simpler because the correction theorem is applied to scalar functions rather than to vector-valued ones. The reader may wish to consult [2] for a motivation and more detailed discussion of this problematic.

THEOREM. *If $d \geq 3$ there is a harmonic function $f : \mathbf{R}_+^d \to \mathbf{R}$ which is C^1 up to the boundary and such that f and ∇f vanish on a common boundary set with positive measure.*

Outline of proof is as follows. Start with a function u_0 which vanishes on an open subset of the boundary. Using a successive modification ("correction theorem") procedure, decrease the normal derivative to 0 on a subset with large measure. To get the Theorem it is necessary to keep the original function unchanged on a set with large measure. Thus the functions added on during the modification procedure must have small compact support. The idea for constructing those functions is due to A. B. Aleksandrov–P. Kargaev [1] who used it for a closely related problem.

LEMMA 1. *If $p > 0$ is small enough then for all sufficiently small ε there is $F_\varepsilon : \mathbf{R}^{d-1} \to \mathbf{R}$ with $\operatorname{supp} F_\varepsilon \subset D(0, \varepsilon^{1/2})$ $(= \{x \in \mathbf{R}^{d-1} : |x| \leq \varepsilon^{1/2}\})$ and (letting $\widehat{F}_\varepsilon$ be its harmonic extension to $\mathbf{R}_+^d$)*

$$\int_{\mathbf{R}^{d-1}} (|1 + d\widehat{F}_\varepsilon/dn|^p - 1)\, dx < -\eta$$

with $\eta > 0$ independent of ε. Moreover, $|\nabla \widehat{F}_\varepsilon| \lesssim \min(\varepsilon^{-d}, |x|^{-d})$ on $\mathbf{R}^{d-1}$.

Proof. Aleksandrov–Kargaev use the function $G_\varepsilon : \mathbf{R}^d_+ \to \mathbf{R}$ defined by $G_\varepsilon(x) = -(\varepsilon + x_d)/|x + \varepsilon e_d|^d$ (e_d = the unit vector in the x_d direction) which, as they show, satisfies $\int(|1 + dG_\varepsilon/dn|^p - 1)\,dx < -2M < 0$ for p below a certain critical number and ε sufficiently small. One can go from this function to F_ε as above, as follows:

Let $\delta = \varepsilon^{1/2}$ and for $j = 0,1,2,\dots$ let $\psi_j : \mathbf{R}^{d-1} \to \mathbf{R}$ be functions with $\operatorname{supp}\psi_j \subset \{2^{j-1}\delta \le |x| \le 2^{j+1}\delta\}$ and $\sum \psi_j = 1$ for $|x| > \delta$, and with the natural bounds, i.e. $|\nabla^k \psi_j| \lesssim (2^j\delta)^{-k}$. Let $\psi = \sum \psi_j$. Let $\varrho^j_\varepsilon = \psi_j G_\varepsilon$, $\varrho_\varepsilon = \sum_j \varrho^j_\varepsilon = \psi G_\varepsilon$. Let $F_\varepsilon = G_\varepsilon - \varrho_\varepsilon$. To make the estimates, let ∇_T be differentiation in the $\mathbf{R}^{d-1}$ directions. Then $|\nabla^k_T G_\varepsilon| \le \varepsilon|x|^{-(d+k)}$, $k = 0,1,2$. Therefore $|\nabla^k_T \varrho^j_\varepsilon| \lesssim \varepsilon(2^j\delta)^{-(d+k)}$. This implies $\|\nabla_T \varrho^j_\varepsilon\|_1 \lesssim \varepsilon(2^j\delta)^{-2}$, and also $\|\nabla_T \varrho^j_\varepsilon\|_{C^\alpha} \lesssim \varepsilon(2^j\delta)^{-(d+1+\alpha)}$. By $L^1 \to$ weak-L^1 and Hölder estimates for the Riesz transforms we have

$$|\{x : |(d\widehat{\varrho^j_\varepsilon}/dn)(x)| > \lambda\}| < \varepsilon(2^j\delta)^{-2}\lambda^{-1}\,,$$
$$\|d\widehat{\varrho^j_\varepsilon}/dn\|_{C^\alpha} \lesssim \varepsilon(2^j\delta)^{-(d+1+\alpha)}\,.$$

It follows that $|(d\widehat{\varrho^j_\varepsilon}/dn)(x)\| \lesssim \varepsilon(2^j\delta)^{-(d+1)}$ when $|x| < 2\cdot 2^{j+1}\delta$ (the weak type 1 estimate implies this holds for *some* x with $|x| < 2\cdot 2^{j+1}\delta$ and the Hölder estimate extends it to *all* such x). Furthermore, if $|x| > 2\cdot 2^{j+1}\delta$ then

$$\left|\frac{d\widehat{\varrho^j_\varepsilon}}{dn}(x)\right| = \left|\text{const}\cdot \int_{|y|<2^{j+1}\delta} |x-y|^{-d}\varrho^j_\varepsilon(y)\,dy\right| \lesssim \varepsilon(2^j\delta)^{-1}|x|^{-d}\,,$$

using $|\varrho^j_\varepsilon| \lesssim \varepsilon(2^j\delta)^{-d}$. In other words, we have $|d\widehat{\varrho^j_\varepsilon}/dn| \lesssim \varepsilon(2^j\delta)^{-1} \times \min(|x|^{-d}, (2^j\delta)^{-d})$ on $\mathbf{R}^{d-1}$. Summing over j we obtain

$$(*)\qquad |d\widehat{\varrho}_\varepsilon/dn| \lesssim \varepsilon\delta^{-1}\min(\delta^{-d}, |x|^{-d})\,.$$

Then for small p and appropriate η

$$\begin{aligned}
\int \left|1 + \frac{d\widehat{F}_\varepsilon}{dn}\right|^p - 1
&= \left(\int \left|1 + \frac{dG_\varepsilon}{dn}\right|^p - 1\right) + \int \left(\left|1 + \frac{d\widehat{F}_\varepsilon}{dn}\right|^p - \left|1 + \frac{dG_\varepsilon}{dn}\right|^p\right)\\
&\le -2\eta + \int_{|x|<1} \left(\left|1 + \frac{d\widehat{F}_\varepsilon}{dn}\right|^p - \left|1 + \frac{dG_\varepsilon}{dn}\right|^p\right)\\
&\quad + \int_{|x|>1} \left(\left|1 + \frac{d\widehat{F}_\varepsilon}{dn}\right|^p - \left|1 + \frac{dG_\varepsilon}{dn}\right|^p\right).
\end{aligned}$$

Now

$$\int_{|x|<1} \left(\left|1 + \frac{d\widehat{F}_\varepsilon}{dn}\right|^p - \left|1 + \frac{dG_\varepsilon}{dn}\right|^p \right) \le \int_{|x|\le 1} \left|\frac{d\varrho_\varepsilon}{dn}\right|^p$$

($(a+b)^p \le a^p + b^p$ when $p < 1$) and $(*)$ implies this goes to 0 with ε. When $|x| > 1$ we have a lower bound for $|1 + dG_\varepsilon/dn|$, and $|d\varrho_\varepsilon/dn|$ is small. It follows by the mean value theorem that $|1 + d\widehat{F}_\varepsilon/dn|^p - |1 + dG_\varepsilon/dn|^p \le C|d\varrho_\varepsilon/dn|$ and therefore that

$$\int_{|x|>1} \left(\left|1 + \frac{d\widehat{F}_\varepsilon}{dn}\right|^p - \left|1 + \frac{dG_\varepsilon}{dn}\right|^p \right) \le C \int_{|x|>1} \left|\frac{d\varrho_\varepsilon}{dn}\right| \le C\varepsilon\delta^{-1},$$

which again goes to 0 with ε. That proves $\int |1 + d\widehat{F}_\varepsilon/dn|^p - 1 < -\eta$.

To prove $|\nabla \widehat{F}_\varepsilon| \lesssim \min(\varepsilon^{-d}, |x|^{-d})$: this estimate is obvious for $|\nabla G_\varepsilon|$ and for $|\nabla_T \varrho_\varepsilon|$. For $|d\widehat{\varrho}_\varepsilon/dn|$ it follows from $(*)$. ∎

LEMMA 2. *If N is large enough there is a constant $\beta = \beta(N) > 0$ such that if $Q \subset \mathbf{R}^{d-1}$ is a cube, a_Q its center and $I : Q \to \mathbf{R}$ is a function such that $N^{d-1}|I(a_Q)|^{-1} \sup_{x\in Q} |I(x) - I(a_Q)|$ is sufficiently small (independently of N and ε) then*

$$\left(\int_Q \left| I(x) + \frac{d\widehat{F}_\varepsilon}{dn}(Nl(Q)^{-1}(x - a_Q)) \right|^p dx \right)^{1/p} < e^{-2\beta} |I(a_Q)|\, |Q|^{1/p}.$$

Proof. We can assume $Q = Q(N) = [-N/2, N/2] \times \ldots \times [-N/2, N/2]$. It is then equivalent to show that if $N^{d-1}\|I - 1\|_\infty$ is sufficiently small then

$$\int_{Q(N)} \left(\left| I + \frac{d\widehat{F}_\varepsilon}{dn} \right|^p - 1 \right) \le -\eta$$

for suitable $\eta > 0$. This last inequality is true since

$$\begin{aligned}
&\int_{Q(N)} \left(\left|1 + \frac{d\widehat{F}_\varepsilon}{dn}\right|^p - \left|1 + \frac{d\widehat{F}_\varepsilon}{dn}\right|^p \right) dx \\
&\qquad \le \int_{Q(N)\setminus D(0,1)} + \int_{D(0,1)} \\
&\qquad \le C \int_{Q(N)\setminus D(0,1)} |I - 1|\, dx + \int_{D(0,1)} |I - 1|^p\, dx
\end{aligned}$$

(because $1 + dF_\varepsilon/dn$ is bounded away from 0 when $|x| > 1$)

$$\le C\|I - 1\|_\infty N^{d-1} + C\|I - 1\|_\infty,$$

which is small. It remains to apply Lemma 1. ∎

Now we give the recursive construction. For a sufficiently large N to be determined, choose β according to Lemma 2. Also fix sequences $K_n \to \infty$, $\varepsilon_n \searrow 0$ (to be determined). Constants denoted by C will be independent of these choices. Let u_0 be a smooth function on $\mathbf{R}^{d-1}$ which vanishes on $Q(1)$.

We will drop the $\hat{}$ notation and identify functions on $\mathbf{R}^{d-1}$ with their harmonic extensions when there is no confusion.

If the nth stage of the construction has been done ($n \geq 0$) we will have the following data: a number $\delta_n > 0$ with $\delta_n^{-1} \in \mathbf{Z}$, a subset $\mathcal{G}_n \subset \mathcal{H}_n$ where $\mathcal{H}_n$ is the collection of $\delta_n^{-(d-1)}$ cubes of side δ_n whose union is $Q(1)$, and a smooth function u_n such that

$$\left(\int_{V_n} \left| \frac{du_n}{dn} \right|^p \right)^{1/p} \leq Ae^{-\beta n}$$

where $V_n = \bigcup \{Q : Q \in \mathcal{G}_n\}$, β is as above and A is a sufficiently large constant which is independent of n. To start take $\delta_0 = 1$, $\mathcal{G}_0 = \{Q(1)\}$.

To do stage $n+1$ choose δ_{n+1} with $\delta_n/\delta_{n+1} \in \mathbf{Z}$ such that δ_{n+1} is extremely small—how small will be determined later. $\mathcal{G}_{n+1}$ is then all cubes $Q \in \mathcal{H}_{n+1}$ such that

(a) the cube $Q' \in \mathcal{H}_n$ with $Q \subset Q'$ satisfies $Q' \in \mathcal{G}_n$, and

(b) $(|Q|^{-1} \int_Q |du_n/dn|^p)^{1/p} < K_{n+1}e^{-\beta n}$.

Let a_Q be the center of Q and define

$$u_{n+1}(x) = u_n(x) + \sum_{Q \in \mathcal{G}_{n+1}} \frac{du_n}{dn}(a_Q) F_{\varepsilon_{n+1}}(N\delta_{n+1}^{-1}(x - a_Q)) \frac{\delta_{n+1}}{N}$$

so that

$$\nabla u_{n+1}(x) = \nabla u_n(x) + \sum_{Q \in \mathcal{G}_{n+1}} \frac{du_n}{dn}(a_Q) \nabla F_{\varepsilon_{n+1}}(N\delta_{n+1}^{-1}(x - a_Q)).$$

LEMMA 3.

$$\sum_{Q \in \mathcal{G}_{n+1} : |x - a_Q| > \varrho} |\nabla F_{\varepsilon_{n+1}}(N\delta_{n+1}^{-1}(x - a_Q))| \leq CN^{-d}\delta_{n+1}\varrho^{-1}$$

for all $\varrho >$ const $\cdot\, \delta_{n+1}$ and $x \in \mathbf{R}^{d-1}$ (C depends on a lower bound for ϱ/δ_{n+1}).

Proof. The left hand side is $\leq \sum_{|x-a_Q|>\varrho} (N\delta_{n+1}^{-1}|x - a_Q|)^{-d}$ by the last part of Lemma 1. Lemma 3 follows by considering $\sum \delta_{n+1}^{d-1}|x - a_Q|^{-d}$ as Riemann sum for $\int_{|x|>\varrho} |x|^{-d}\,dx$, which is justified as long as ϱ/δ_{n+1} does not approach 0.

COROLLARY. *If* δ_{n+1} *is small enough, then on* $\mathbf{R}^{d-1}$,

$$|\nabla u_{n+1} - \nabla u_n| \le C K_{n+1} e^{-\beta n} \varepsilon_{n+1}^{-d}.$$

Proof. Assume for notational purposes that $x \in Q \in \mathcal{G}_{n+1}$; the other case is a little simpler. If δ_{n+1} is small, then by (b) and smoothness of u_n, we know

$$|\nabla u_n(a_{Q'})| \le 2K_{n+1} e^{-\beta n}$$

for all $Q' \in \mathcal{G}_{n+1}$. So

$$\begin{aligned}
|\nabla u_{n+1}(x) - \nabla u_n(x)| &\le \sum_{\substack{Q' \in \mathcal{G}_{n+1} \\ Q' \ne Q}} |\nabla u_n(a_{Q'})| |\nabla F_{\varepsilon_{n+1}}(N\delta_{n+1}^{-1}(x - a_{Q'}))| \\
&\quad + |\nabla u_n(a_Q)| |\nabla F_{\varepsilon_{n+1}}(N\delta_{n+1}^{-1}(x - a_Q))| \\
&\le 2K_{n+1} e^{-\beta n} \Big[\sum_{Q' \ne Q} |\nabla F_{\varepsilon_{n+1}}(N\delta_{n+1}^{-1}(x - a_{Q'}))| \\
&\quad + |\nabla F_{\varepsilon_{n+1}}(N\delta_{n+1}^{-1}(x - a_Q))| \Big] \\
&\le 2K_{n+1} e^{-\beta n}(C + C\varepsilon_{n+1}^{-d})
\end{aligned}$$

where the sum was estimated by Lemma 3 and the other term by Lemma 1. ∎

Next we verify the induction hypothesis at stage $n+1$, i.e.

LEMMA 4. *If* A *is large then* $(\int_{V_{n+1}} |du_{n+1}/dn|^p)^{1/p} < Ae^{-\beta(n+1)}$.

Proof. First we show the following statement: suppose $\gamma > 0$ is given. Then if δ_{n+1} is small enough we will have for any $x \in Q \in \mathcal{G}_{n+1}$

$$(*) \quad \sum_{\substack{Q' \in \mathcal{G}_{n+1} \\ Q' \ne Q}} \left| \frac{du_n}{dn}(a_{Q'}) \right| |\nabla F_{\varepsilon_{n+1}}(N\delta_{n+1}^{-1}(x - a_{Q'}))| \le CN^{-d} \left| \frac{du_n}{dn}(x) \right| + \gamma.$$

Proof of $(*)$. Fix $M < \infty$ and split the sum into $\sum_{Q' : |x - a_{Q'}| < M\delta_{n+1}}$ and $\sum_{Q' : |x - a_{Q'}| > M\delta_{n+1}}$. Then

$$\begin{aligned}
\sum_{Q' : |x - a_{Q'}| < M\delta_{n+1}} &\le \sup\left(\left| \frac{du_n}{dn}(a_{Q'}) \right| : |x - a_{Q'}| < M\delta_{n+1} \right) \\
&\quad \times \sum_{Q' \ne Q} |\nabla F_{\varepsilon_{n+1}}(N\delta_{n+1}^{-1}(x - a_{Q'}))| \\
&< CN^{-d} \sup\left(\left| \frac{du_n}{dn}(a_{Q'}) \right| : |x - a_{Q'}| < M\delta_{n+1} \right) \\
&\qquad \text{(by Lemma 3 with } \varrho = \tfrac{1}{2}\delta_{n+1})
\end{aligned}$$

$$< CN^{-d}\left(\left|\frac{du_n}{dn}(x)\right| + \gamma\right)$$

where $\gamma \to 0$ as $\delta_n \to 0$ by continuity of $|du_n/dn|$. Also

$$\begin{aligned}\sum_{Q':|x-a_{Q'}|>M\delta_{n+1}} &\le \sup\left(\left|\frac{du_n}{dn}(y)\right| : y \in V_{n+1}\right) \\ &\quad \times \sum_{|x-a_{Q'}|>M\delta_{n+1}} |\nabla F_{\varepsilon_{n+1}}(N\delta_{n+1}^{-1}(x-a_{Q'}))| \\ &\le 2K_{n+1}e^{-\beta n} \sum_{|x-a_{Q'}|>M\delta_{n+1}} |\nabla F_{\varepsilon_{n+1}}(N\delta_{n+1}^{-1}(x-a_{Q}))| \\ &\qquad\qquad (\text{provided } \delta_{n+1} \text{ is small enough}) \\ &\le CK_{n+1}e^{-\beta n}\cdot M^{-1}\end{aligned}$$

by Lemma 3 ($\rho = M\delta_{n+1}$). Since M is arbitrary we can make this as small as we want, which proves $(*)$.

Now, we will call $Q \in \mathcal{G}_{n+1}$ *type* 1 if $|(du_n/dn)(a_Q)| > e^{-4\beta(n+1)}$ and *type* 2 otherwise. Fix $Q \in \mathcal{G}_{n+1}$. Write (for $x \in Q$)

$$\begin{aligned}\frac{du_{n+1}}{dn}(x) = \frac{du_n}{dn}(a_Q) &+ \left[\frac{du_n}{dn}(x) - \frac{du_n}{dn}(a_Q)\right. \\ &+ \left.\sum_{\substack{Q'\ne Q\\ Q'\in\mathcal{G}_{n+1}}} \frac{du_n}{dn}(a_{Q'})\frac{dF_{\varepsilon_{n+1}}}{dn}(N\delta_{n+1}^{-1}(x-a_{Q'}))\right] \\ &+ \frac{du_n}{dn}(a_Q)\frac{dF_{\varepsilon_{n+1}}}{dn}(N\delta_{n+1}^{-1}(x-a_Q)).\end{aligned}$$

By $(*)$ and smoothness of du_n/dn the term in brackets may be made less than $CN^{-d}|(du_n/dn)(a_Q)| + \gamma$ with γ arbitrarily small. For Q type 1, the bracketed term is therefore $< CN^{-d}|(du_n/dn)(a_Q)|$ provided γ is chosen right. So Lemma 2 applies (we are using here the fact that $CN^{-d} < cN^{-(d-1)}$ for large N) and we get

$$\left(\int_Q \left|\frac{du_{n+1}}{dn}\right|^p\right)^{1/p} < e^{-2\beta}\left|\frac{du_n}{dn}(a_Q)\right||Q|^{1/p} < e^{-3\beta/2}\left(\int_Q \left|\frac{du_n}{dn}\right|^p\right)^{1/p}$$

if δ_{n+1} is small. If Q is type 2 the bracketed term is $< CN^{-d}|(du_n/dn)(a_Q)| + \gamma$ which may be taken $< e^{-4\beta(n+1)}$. So,

$$\begin{aligned}\int_Q \left|\frac{du_{n+1}}{dn}\right|^p &\le \left|\frac{du_{n+1}}{dn}(a_Q)\right|^p \int_Q \left|1 + \frac{dF_{\varepsilon_{n+1}}}{dn}(N\delta_{n+1}^{-1}(x-a_Q))\right|^p + \int_Q [\ ]^p \\ &\le 2e^{-4\beta(n+1)p}|Q|\end{aligned}$$

since (say) the first integral is $< |Q|$ by Lemma 2. We conclude

$$\begin{aligned}\int_{V_{n+1}} \left|\frac{du_{n+1}}{dn}\right|^p &= \int_{U\{Q \text{ type } 1\}} + \int_{U\{Q \text{ type } 2\}} \\ &\le e^{-3p\beta/2} \int_{U\{Q \text{ type } 1\}} \left|\frac{du_n}{dn}\right|^p + Ce^{-4p\beta(n+1)} \sum_{Q \text{ type } 2} |Q| \\ &\le A^p e^{-p\beta(n+1)} \cdot e^{-p\beta/2} + Ce^{-4p\beta(n+1)} \\ &= A^p e^{-p\beta(n+1)}(e^{-p\beta/2} + CA^{-p}e^{-3p\beta(n+1)}).\end{aligned}$$

The parenthetical term is < 1 if A was large enough. ∎

Now we give the conditions on the K_n and ε_n: they should satisfy

(1) $\sum \varepsilon_{n+1}^{-d} K_{n+1} e^{-\beta n} < \infty$,

(2) $\sum K_{n+1}^{-p} + \varepsilon_{n+1}^{(d-1)/2}$ is sufficiently small.

These are clearly compatible, e.g. we can take $\varepsilon_n = C^{-1}n^{-2}$, $K_n = Cn^{2/p}$ for a suitable large constant C.

Condition (1) implies by the Corollary to Lemma 2 that the $\widehat{u}_n$ converge in C^1 norm on the closed upper half space. Call the limit function f. We have to show $f = df/dn = 0$ on a set of positive measure. But

$$|\{x \in Q(1) : f(x) \ne 0\}| \le \sum_n |\{x \in Q(1) : u_{n+1}(x) \ne u_n(x)\}|$$

and $\{x : u_{n+1}(x) \ne u_n(x)\}$ is a union of at most $\delta_{n+1}^{-(d-1)}$ discs each of radius $\delta_{n+1}\varepsilon_{n+1}^{1/2}$ (because of $\operatorname{supp} F_\varepsilon$ being contained in $D(0, \varepsilon^{1/2})$). So,

$$|\{x \in Q(1) : f(x) \ne 0\}| \le C \sum_n \delta_{n+1}^{-(d-1)} (\delta_{n+1}\varepsilon_{n+1}^{1/2})^{d-1} \le C \sum \varepsilon_{n+1}^{(d-1)/2}.$$

Also, $d\widehat{f}/dn = 0$ on $\bigcap_n V_n$. Thus

$$\left|\left\{x \in Q(1) : \frac{d\widehat{f}}{dn}(x) \ne 0\right\}\right| \le \sum_n |V_n \setminus V_{n+1}|.$$

What is the measure of $V_n \setminus V_{n+1}$? If $Q \notin \mathcal{G}_{n+1}$ but $Q \subset Q' \in \mathcal{G}_n$, then

$$|Q|^{-1} \int_Q |du_n/dn|^p > K_{n+1}^p e^{-p\beta n}, \quad \text{i.e.} \quad |Q| < K_{n+1}^{-p} e^{-\beta pn} \int_Q |du_n/dn|^p.$$

So,

$$|V_n \setminus V_{n+1}| \le K_{n+1}^{-p} e^{\beta pn} \int_{V_n} \left|\frac{du_n}{dn}\right|^p \le A^p K_{n+1}^{-p}$$

and $|\{x \in Q(1) : (d\widehat{f}/dn)(x) \ne 0\}| \le A^p \sum K_{n+1}^{-p}$. That means $|\{x \in Q(1) : (d\widehat{f}/dn)(x) \ne 0\}| + |\{x \in Q(1) : f(x) \ne 0\}| \le C \sum \varepsilon_{n+1}^{(d-1)/2} + A^p \sum K_{n+1}^{-p} < 1$

if (2) holds. So $Q(1) \cap \{x : (d\hat{f}/dn)(x) = 0\} \cap \{x : f(x) = 0\}$ must have positive measure.

Remarks. 1) To minimize technicalities we did not try to obtain a Hölder estimate on the gradient although this should be possible along the lines of [2].

2) The main question remains open, that is, whether it is possible to have harmonic functions C^2 or C^∞ smooth up to the boundary whose gradients vanish on sets of positive measure.

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INSTITUT DES HAUTES ETUDES SCIENTIFIQUES
35, ROUTE DE CHARTRES
91440 BURES-SUR-YVETTE, FRANCE

DEPARTMENT OF MATHEMATICS
CALIFORNIA INSTITUTE
OF TECHNOLOGY
PASADENA, CALIFORNIA 91125 U.S.A

Reçu par la Rédaction le 28.2.1990