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ON MARTIN'S AXIOM AND PERFECT SPACES

BY

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We show that Martin's Axiom (MA), together with relatively mild conditions imposed on a space X, implies that X is perfect provided that X is a union of less than continuum compact subsets. This leads to a strengthening of Theorem 1 from [1].

Assuming $\mathrm{MA} + \neg\mathrm{CH}$ we construct an example of a locally compact separable Moore space which is countably paracompact but not normal. This example answers one of the questions asked by Juhász and Weiss in [6].

A space is *submetrizable* if it has a weaker metric topology. A space (X, τ) is *cometrizable* ([1], see also [9] and [3]) if it has a weaker metric separable topology μ such that τ is regular with respect to μ (i.e. if for every $x \in U \in \tau$ there exists a $V \in \tau$ such that $x \in V \subset \mathrm{Cl}_\mu V \subset U$).

A space is *perfect* if its closed subsets are G_δ-sets. A space is *perfectly normal* if it is normal and perfect. A space X has a *G_δ-diagonal* (*regular G_δ-diagonal*) if there exists a countable family $\{G_n\}_{n<\omega}$ of neighbourhoods of the diagonal $\Delta = \{(x, x) : x \in X\} \subset X^2$ such that

$$\Delta = \bigcap_{n<\omega} G_n \quad \Big(\Delta = \bigcap_{n<\omega} \bar{G}_n\Big).$$

A family $\mathcal{U}$ of open subsets of X is called a *pseudobase* (*regular pseudobase*) if for every $x \in X$ we have

$$\{x\} = \bigcap \{U : x \in U \in \mathcal{U}\} \quad \Big(\{x\} = \bigcap \{\bar{U} : x \in U \in \mathcal{U}\}\Big).$$

We say that X is a *small-size space* if it is the union of less than 2^ω compact subsets, in particular, if its cardinality $|X|$ is less than 2^ω.

THEOREM 1. (MA) *If a small-size X has a countable regular pseudobase, then X^ω is perfect.*

THEOREM 2. (MA) *If a space X of cardinality less than 2^ω has a countable pseudobase, then all subsets of X are G_δ-sets and the space X^ω is perfect.*

In Theorem 2, which is a slight generalization of a result due to Silver (cf. [14], Lemma 3), it suffices to assume that X has a σ-point finite pseudobase (see Remark 3). Before proving Theorems 1 and 2 let us derive some corollaries to them.

COROLLARY 1. (MA) *If a small-size X is submetrizable, then X^ω is perfect.*

Proof. Since every submetrizable compact space is metrizable, and thus has cardinality less than or equal to 2^ω, the space X has cardinality less than or equal to 2^ω. A countable base for the weaker metric separable topology on X ([9], Proposition 1) is a countable regular pseudobase for X.

The following corollary is a strengthening of Theorem 1 from [1]:

COROLLARY 2. (MA) *If a small-size X is cometrizable, then X^ω is perfectly normal.*.

Proof. By Corollary 1, X^ω is perfect and, by Theorem 1 in [1], X^n is normal for every $n < \omega$. Thus, by Katětov's result [7], X^ω is perfectly normal.

COROLLARY 3. (MA) *If a subparacompact space of cardinality less than 2^ω has a G_δ-diagonal, then all subsets of X are G_δ-sets and the space X^ω is perfect.*

Proof. It is proved in [12] that every subparacompact space with a G_δ-diagonal and cardinality less than or equal to 2^ω has a countable pseudobase.

COROLLARY 4. (MA) *Every subset of a semistratifiable space of cardinality less than 2^ω is a G_δ.*

COROLLARY 5. (MA) *Every subset of a Moore space of cardinality less than 2^ω is a G_δ.*

Remark 1. Reed [10] has obtained Corollary 5 under the additional assumption that the Moore space is normal, as a consequence of the fact that normal Moore spaces of cardinality less than or equal to 2^ω are submetrizable [12]. The Example shows, however, that there exist even locally compact, separable Moore spaces of cardinality ω_1 which are neither normal nor submetrizable and which also do not have countable regular pseudobases (see also [11]).

COROLLARY 6. (MA) *If X is a small-size paracompact space, then X^ω is perfect iff X has a G_δ-diagonal.*

Proof. Borges [2] and Okuyama [8] proved that a paracompact space with a G_δ-diagonal is submetrizable.

COROLLARY 7. (MA) *If X is a small-size separable space with a regular G_δ-diagonal, then X^ω is perfect.*

Proof. Every separable space with a regular G_δ-diagonal has a countable regular pseudobase ([6], Lemma 2.6).

Remark 2. It is easy to verify that all the above-stated results are actually independent of the ZFC axioms of set theory. Also, almost all of the assumptions appearing in their statements are essential. The following problem, however, seems to be open:

PROBLEM 1. Does MA imply that every small-size space with a countable pseudobase or a G_δ-diagonal is perfect? (**P 1203**)

The proofs of Theorems 1 and 2 are based on the following set-theoretic proposition, which is known to be a consequence of Martin's Axiom (see, e.g., [13]):

(S) *Let $\mathscr{A}$ and $\mathscr{B}$ be families of cardinality less than 2^ω of subsets of ω such that $A\setminus\bigcup\mathscr{C}$ is infinite for every $A\in\mathscr{A}$ and for a finite subfamily $\mathscr{C}$ of $\mathscr{B}$. Then there exists a subset E of ω such that $A\cap E$ is infinite for every $A\in\mathscr{A}$, and $B\cap E$ is finite for every $B\in\mathscr{B}$.*

Proof of Theorem 1 (cf. [14]). Let $\mathscr{C}$ be a family of cardinality less than 2^ω of compact subsets of X such that $X=\bigcup\mathscr{C}$. Since X has a countable regular pseudobase, there exists a countable family $\mathscr{G}=\{G_n\}_{n<\omega}$ of open subsets of X such that, for any two disjoint compact subsets K and L of X, there exists an $n<\omega$ for which $K\subset G_n\subset X\setminus L$.

We can assume that every element G of $\mathscr{G}$ appears in the sequence $\{G_n\}_{n<\omega}$ infinitely many times.

The existence of $\mathscr{G}$ implies easily that every compact subspace C of X has a countable base, and thus is metrizable.

It is enough to prove that X is perfect, because all finite products X^n of X also are of small-size and have countable regular pseudobases, so that it suffices to recall that if X^n is perfect for every $n<\omega$, then X^ω is perfect [5].

Let F be a closed subset of X and let

$$\mathscr{K}=\{F\cap C: C\in\mathscr{C}\}.$$

Since for every $C\in\mathscr{C}$ the set $C\setminus F$ is an F_σ in C and hence in X, there exists a family $\mathscr{L}$ of cardinality less than 2^ω of compact subsets of X such that $X\setminus F=\bigcup\mathscr{L}$. Thus we have

$$\bigcup\mathscr{K}=F,\qquad \bigcup\mathscr{K}\cup\bigcup\mathscr{L}=X,\qquad \bigcup\mathscr{K}\cap\bigcup\mathscr{L}=\emptyset.$$

For every $K\in\mathscr{K}$ let $A_K=\{n<\omega: K\subset G_n\}$ and for every $L\in\mathscr{L}$ let $B_L=\{n<\omega: L\cap G_n\neq\emptyset\}$. Put

$$\mathscr{A}=\{A_K: K\in\mathscr{K}\}\quad\text{and}\quad\mathscr{B}=\{B_L: L\in\mathscr{L}\}.$$

Clearly, the families $\mathscr{A}$ and $\mathscr{B}$ of subsets of ω have cardinality less than 2^ω. Let us choose a $K\in\mathscr{K}$ and a finite subfamily $L_1, L_2, \ldots, L_m$ of $\mathscr{L}$. The set

$$D=A_K\setminus\bigcup_{i=1}^{m}B_{L_i}$$

is infinite. Indeed, since the set

$$L = \bigcup_{i=1}^{m} L_i$$

is compact and disjoint from K, there exist infinitely many $n < \omega$ such that $K \subset G_n \subset X \setminus L$. Any such n belongs to D.

By (S) there exists a subset E of ω such that $A_K \cap E$ is infinite for every $K \in \mathscr{K}$ and $B_L \cap E$ is finite for every $L \in \mathscr{L}$. Let us put

$$U_m = \bigcup \{G_n : n \in E \text{ and } n > m\}.$$

It suffices to show that

$$F = \bigcap_{m<\omega} U_m.$$

For every $m < \omega$ and $K \in \mathscr{K}$ there exists an $n > m$ with $n \in A_K \cap E$, which implies $K \subset G_n \subset U_m$ and $F \subset U_m$. On the other hand, for every $L \in \mathscr{L}$ there exists an $n < \omega$ such that $B_L \cap E$ contains no $n > m$. Thus

$$L \cap U_M = \emptyset \quad \text{and} \quad \bigcap_{m<\omega} U_m \subset F,$$

which completes the proof.

Proof of Theorem 2. This proof is completely analogous to the previous one, so we only sketch it (see also [14], Lemma 3).

Let X be a space of cardinality less than 2^ω with a countable pseudobase $\mathscr{G} = \{G_n\}_{n<\omega}$. We can clearly assume that $\mathscr{G}$ is closed under finite intersections and that every element G of $\mathscr{G}$ appears in the sequence $\{G_n\}_{n<\omega}$ infinitely many times.

Again, it suffices to prove that every subset F of X is a G_δ. For every $x \in F$ and $y \in X \setminus F$ put

$$A_x = \{n < \omega : x \in G_n\} \quad \text{and} \quad B_y = \{n < \omega : y \in G_n\}.$$

The families $\mathscr{A} = \{A_x\}_{x \in F}$ and $\mathscr{B} = \{B_y\}_{y \in X \setminus F}$ clearly satisfy conditions of (S). Take $E \subset \omega$ whose existence is assured by (S) and put

$$U_m = \bigcup \{G_n : n \in E \text{ and } n > m\}.$$

One easily verifies that

$$F = \bigcap_{m>\omega} U_m.$$

Remark 3. After the results of this paper had been completed the author learnt of a joint paper [4] by Fleissner and Reed and noticed that Theorem 3.1 from their paper follows immediately from our Theorem 2 and

PROPOSITION. *A space X of cardinality less than or equal to 2^ω has a countable pseudobase if and only if it has a σ-point finite pseudobase.*

Proof. Let

$$\mathscr{B} = \bigcup_{n<\omega} \mathscr{B}_n$$

be a pseudobase for X such that all families $\mathscr{B}_n$ are point finite. Let $f: \mathscr{B} \to R$ be an arbitrary one-to-one function into the real line R and let $\{U_m\}_{m<\omega}$ be a countable base for R closed with respect to finite unions. For every $n, m < \omega$ put

$$V_{n,m} = \bigcup \{B \in \mathscr{B}_n : f(B) \notin U_m\}.$$

It suffices to show that the family $\{V_{n,m}\}_{n,m<\omega}$ is a pseudobase for X.

Let $x, y \in X$ and $x \neq y$. There exist an $n < \omega$ and a $\hat{B} \in \mathscr{B}_n$ such that $x \in \hat{B} \subset X \setminus \{y\}$. Let $\mathscr{B}^* = \{B \in \mathscr{B}_n : y \in B\}$. There exists an $m < \omega$ such that $f(\mathscr{B}^*) \subset U_m \subset R \setminus \{f(\hat{B})\}$. Clearly, $x \in V_{n,m} \subset X \setminus \{y\}$.

Example. A locally compact separable Moore space X with a countable pseudobase which is neither normal nor submetrizable and does not have a countable regular pseudobase nor a regular G_δ-diagonal.

MA + $\neg$CH implies that X is a small-size countably paracompact space.

The Example answers a question asked by Juhász and Weiss ([6], a remark following Lemma 2.5) whether every locally compact Moore space of cardinality less than or equal to 2^ω has a countable regular pseudobase. Recall that every Moore space of cardinality less than or equal to 2^ω has a countable pseudobase [12].

The Example seems to be also the first known example of a locally compact countably paracompact non-normal Moore space (cf. [11]) and leads to the following

PROBLEM 2. Is the existence of a locally compact countably paracompact non-normal Moore space independent of the axioms of set theory (**P 1204**)

Let us recall that the existence of a normal locally compact non-metrizable Moore space is independent of the axioms of set theory.

Construction of the Example. We use the splitting technique from [11]. Let C be the Cantor set and let Δ be the diagonal $\{(z, z) : z \in C\}$ of C^2. Choose a subset E of C of cardinality ω_1 and let E_1 and E_2 be two disjoint copies of E which are also disjoint from C^2 and assume that if $z \in E$, then z_1 and z_2 are counterparts of z in E_1 and E_2, respectively. Let $X = C^2 \setminus \Delta \cup E_1 \cup E_2$. We generate a topology on X. Points $(x, y) \in C^2 \setminus \Delta$ will have usual neighbourhoods inherited from C^2.

For $z \in E$ put

$$A(z) = \{(z, y) : y \in C \text{ and } y > z\},$$

$$B(z) = \{(x, z) : x \in C \text{ and } x < z\},$$

$$U(z) = \{(x, y) \in C^2 : y > 2x - z \text{ and } y > -2x + 3z\},$$

$$V(z) = \left\{(x, y) \in C^2 : \frac{1}{2}x + \frac{1}{2}z < y < -\frac{1}{2}x + \frac{3}{2}z\right\}.$$

One easily sees that

$$A(z) \subset U(z) \subset C^2 \setminus \Delta \quad \text{and} \quad B(z) \subset V(z) \subset C^2 \setminus \Delta$$

and that $A(z)$ and $B(z)$ are closed subsets of $C^2 \setminus \Delta$, whilst $U(z)$ and $V(z)$ are open subsets of $C^2 \setminus \Delta$. Since $C^2 \setminus \Delta$ is a zero-dimensional metric separable space, there exist open and closed subsets $G(z)$ and $H(z)$ of $C^2 \setminus \Delta$ such that

$$A(z) \subset G(z) \subset U(z) \quad \text{and} \quad B(z) \subset H(z) \subset V(z).$$

For every $z \in E$ let $K_n(z)$ denote a ball in C^2 of radius $1/n$ and center at the point (z, z). Put

$$G_n(z_1) = \{z_1\} \cup (K_n(z) \cap G(z)),$$

$$H_n(z_2) = \{z_2\} \cup (K_n(z) \cap H(z)), \quad A_n(z) = A(z) \cap K_n(z),$$

and take sets $G_n(z_1)$ and $H_n(z_2)$ to be basic neighbourhoods of points $z_1 \in E_1$ and $z_2 \in E_2$, respectively, with n a natural number.

One easily verifies that X is a locally compact separable Moore space, being the union of ω_1 compact subsets. We shall show that X does not have a countable regular pseudobase, from which it will follow (cf. [6]) that X does not have a regular G_δ-diagonal and is not submetrizable. The existence of a countable pseudobase follows from [12] but can also be easily checked directly. It is easy to see that closed disjoint subsets E_1 and E_2 of X cannot be separated by open sets, which implies the non-normality of X (this follows also from [12]).

Assume that the family $\{W_m\}_{m<\omega}$ of open subsets of X is a regular pseudobase. For every $z \in E$ there exist an $m(z)$ and an $n(z)$ such that $z_1 \in G_{n(z)}(z_1) \subset W_{m(z)}$ and $z_2 \notin \overline{W}_{m(z)}$. There exist an n and an m and an uncountable subset E' of E such that $n(z) = n$ and $m(z) = m$ for every $z \in E'$. There exist a $z \in E'$ and a sequence $\{z^k\}_{k<\omega}$ of points of E' converging to z such that $z^k < z$ for every $k < \omega$. Thus

$$W_m \supset \bigcup_{k<\omega} A_n(z^k),$$

which clearly implies that $z_2 \in \overline{W}_m$, a contradiction.

It remains to show that $\mathrm{MA} + \neg\mathrm{CH}$ implies that X is countably paracompact. This, however, follows easily from the fact that under $\mathrm{MA} + \neg\mathrm{CH}$ every subset of E is a relative G_δ.

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