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A NOTE ON ACYCLIC CONTINUA

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1. Introduction. Ward [4, 5] and others have studied and used the inherent order structure of certain arcwise connected continua. The object of these investigations was generally to obtain fixed point theorems. The present paper is concerned mainly with the relationship between the structure of the space and the inherent partial order. In particular, we find necessary and sufficient conditions for the existence of a closed partial order on the space. We use our results to obtain two fixed point theorems and to obtain two characterizations of trees.

2. Preliminaries. In the sequel a *continuum* will be a compact, connected, Hausdorff space, and an *arc* will be a continuum with exactly two non-cutpoints which are the endpoints of the arc. A continuum X is *arcwise connected* in case each pair of distinct points in X is the set of endpoints of some arc contained in X, and X is *acyclic* in case for each pair of points $\{a, b\}$ in X there is at most one arc with endpoints $\{a, b\}$ contained in X. In the following X will be an arcwise connected, acyclic continuum, and if $x, y \in X$, the unique arc with endpoints $\{x, y\}$ will be denoted by $[x, y]$ (or $[y, x]$).

In this paper the term *nest* will mean a collection $\mathfrak{A}$ of subsets of a set X such that if $A, B \in \mathfrak{A}$, then either $A \subset B$ or $B \subset A$.

Let $e \in X$, and let P be the relation defined by $(x, y) \in P$ if and only if $[e, x] \subset [e, y]$. Ward [4, 5] has shown that P is a partial order, and has characterized this order for trees, and other continua. In fact, if X is a tree, then P is closed in the product topology, order dense and has other desirable properties. It is clear that P is always order dense, but P need not be closed. Also note that e is a unique minimal element.

Notation. Let

(i) $xP = \{y \in X \mid (x, y) \in P\}$,

(ii) $Px = \{y \in X \mid (y, x) \in P\}$,

and write

(iii) $x \leqslant y$ iff $(x, y) \in P$, and $x \geqslant y$ iff $(y, x) \in P$.

Then, since $Px = [e, x]$, we see that Px is a closed set. However, xP need not be closed.

In the following P will denote a partial order defined as above with minimal element e.

If $A \subset X$ denote the closure of A by A^*.

3. Continuous and semi-continuous partial orders. A partial order P is called *continuous* in case it is closed in the product topology, and it is called *semi-continuous* in case the sets xP and Px are closed for each x. In this section we investigate the relationships between properties of the space and the continuity or semi-continuity of P.

Remark. We have already noted that Px is closed for each $x \in X$. Furthermore, a theorem of Nachbin [1] implies that if $P = P^*$, then xP is a closed subset of X for each $x \in X$.

PROPOSITION 1. *Suppose xP is closed for each $x \in X$, and suppose $\mathscr{A} = \{[a_\gamma, b_\gamma] \mid \gamma \in A\}$ is a nest of arcs of X. Then there exists an arc $[a, b] \subset X$ such that $\bigcup \mathscr{A} \subset [a, b]$.*

Proof. First we may assume that $a_\gamma = e$ for all γ. Let $B = \{b_\gamma \colon \gamma \in A\}$. Then, since $\mathscr{A}$ is a nest, B is totally ordered, and thus, Theorem 1 of [3] implies that B has a supremum. The general case follows from the above and the observation that either $[a_\gamma, b_\gamma] \subset [e, b_\gamma]$ or $[a_\gamma, b_\gamma] = [e, a_\gamma] \cup [e, b_\gamma]$. Finally, if b is a supremum for B, then $\bigcup \mathscr{A} \subset [e, b]$.

By using Proposition 1 and the results in [2] and [7], we immediately obtain two fixed point theorems.

COROLLARY 1. *If there is an $e \in X$ such that the partial order P with minimal element e is semi-continuous, and if $f \colon X \to X$ is a single-valued continuous function on X into X, then there is a point $x \in X$ such that $f(x) = x$.*

COROLLARY 2. *Suppose that X is either metrizable or that there are arbitrarily small open subsets of X with arcwise connected components. Let $F \colon X \to X$ be a continuous multi-valued function from X into X such that $F(x)$ is finite for each $x \in X$. If there is an $e \in X$ such that the associated partial order P is semi-continuous, then there is an $x \in X$ such that $x \in F(x)$.*

Remark. If xP is closed for each $x \in X$ (or $P = P^*$), then there exists a set $M \subset X$ such that

$$X = \bigcup \{[e, x_m] \colon x_m \in M\},$$

and for each pair $x_m, x_{m'}$ of distinct elements of M, $x_m \nleqslant x_{m'}$ and $x_{m'} \nleqslant x_m$. The set M is the set of maximal elements of X.

We now develop necessary and sufficient conditions that there exists an element $e \in X$ such that the partial order P with minimal element e is continuous. We consider the following conditions:

CONDITION A. *Let* $\mathscr{A} = \{[e, x_\gamma] \mid \gamma \in A\}$ *be a collection of arcs in* X. *Then* $(\bigcup \mathscr{A})^*$ *is arcwise connected.*

Definition. A subset A of X is called *increasing* (with respect to a partial order P) in case $A = \bigcup \{aP \mid a \in A\}$.

CONDITION B. *A space* X *with partial order* P *satisfies condition* B *in case for each* $x \in X$ *and each increasing open set* U *containing* x *there is an increasing open set* W *such that* $x \in W \subset W^* \subseteq U$.

THEOREM 1. *The order* P *in* X *is continuous if and only if it is semi--continuous and conditions* A *and* B *are satisfied.*

Proof. Suppose that $P = P^*$ and that P has minimal element e. Then xP is closed for all $x \in X$. Let $\alpha = \{[e, x_\gamma] \mid \gamma \in A\}$ be a set of arcs in X and let $x_0 \in (\bigcup \alpha)^*$. We must show that $[e, x_0] \subset (\bigcup \alpha)^*$. Suppose there is an element $z \in [e, x_0]$ such that $z \notin (\bigcup \alpha)^*$.

Let $x_1 = \min\big([z, x_0] \cap (\bigcup \alpha)^*\big)$ and let $S = \{x_\beta \mid \beta \in B\}$ be a net in $\bigcup \alpha$ which converges to x_1. Then call a net $\{z_\beta \mid \beta \in B'\}$ a *lower net* for S if and only if B' is a cofinal subset of B and $z_\beta \leqslant x_\beta$ for all $\beta \in B'$. Let $X_1 = \{y \in [e, x_1] \mid y < x_1$ and there is a lower net for S converging to $y\}$. Note that $e \in X_1$. Let $x_2 = \sup X_1$ and let U, V be disjoint open sets such that $x_2 \in V$ and $x_1 \in U$. Pick a $y \in X_1 \cap V$ such that $[y, x_2] \subset V$, and let $\{z_\beta \mid \beta \in B'\}$ be a lower net which converges to y. Then for each $\beta \in B'$ there is a w_β such that $z_\beta < w_\beta < x_\beta$ and $w_\beta \notin U \cup V$. Let w be a cluster point of the net $\{w_\beta \mid \beta \in B'\}$. Since $P = P^*$, we have $y < w < x_1$ and we also have $w \notin U \cup V$. Further $w \notin [y, x_2]$ since $[y, x_2] \subset V$. Thus, $w \in X_1$ and $x_2 < w$. This is a contradiction to the definition of x_2. We conclude that $[e, x_0] \subset (\bigcup \alpha)^*$.

We now show that condition B holds. Suppose $x \in U$, where U is an open increasing set. (We may assume that $x \neq e$ and that $e \notin U$). Let V be an open set containing x such that $V^* \subset U$, and let $W = \bigcup \{yP \mid y \in V\}$. We now show that W is open. Let $\alpha = \big\{[e, z_m] \mid z_m = \max\{z \mid z \in [e, x_m]$ and $[e, z] \cap V = \square\}\big\}$. Then by condition A, $(\bigcup \alpha)^*$ is arcwise connected, and hence, if $W \cap (\bigcup \alpha)^* \neq \square$, we have $V \cap (\bigcup \alpha)^* \neq \square$. But V is an open set disjoint from $\bigcup \alpha$. Thus $W \cap (\bigcup \alpha)^* = \square$, and $W = X/(\bigcup \alpha)^*$ so W is open. From the definition, W is increasing and since U is increasing and $V \subset U$, we have $W \subset U$. Finally if $z \in W^* \setminus W$, there is a net $\{z_\gamma \mid \gamma \in A\}$ in W which converges to z. Thus, there is a net $\{x_\gamma \mid \gamma \in A\}$ with $x_\gamma \leqslant z_\gamma$ and $x_\gamma \in V$. If y is a cluster point of x_γ, then $y \in V^* \subset U$ and since $P = P^*$, we have $y \leqslant z$. Consequently $z \in U$.

Now suppose that there is an $e \in X$ such that conditions A and B hold and such that xP is closed for each $x \in X$. We must show that $P = P^*$. If $(x, y) \notin P$, then $y \notin xP$ and since xP and Py are closed, we can find an open set U such that $xP \subset U$ and such that $Py \cap U = \square$. Set $W = \bigcup \{yP \mid y \in U\}$. Then W is an increasing set and condition A implies that W is open. Now by condition B there is an increasing open set W_1

such that $x \in W_1 \subset W_1^* \subset W$. Thus, there is an open set V containing Py such that $V \cap W_1^* = \Box$. Then $(x, y) \in W_1 \times V$, and if $(s, t) \in W_1 \times V$, we have $s \nleqslant t$ since W_1 is increasing and $W_1 \cap V = \Box$. Therefore $(x, y) \notin P^*$, and thus $P = P^*$.

Remark. In a hereditarily locally connected continuum condition A is always satisfied. However, the example below shows that P need not be semi-continuous for any choice of minimal element, and other examples exist in which condition A holds and in which P is semi-continuous and in which condition B fails.

Before giving the example, we examine briefly the structure of trees and obtain a characterization of trees.

Definitions. A continuum X is a *tree* if each pair of points is separated in X by the omission of a third point. A continuum X is a *local tree* in case each $x \in X$ has arbitrarily small neighborhoods which are trees.

Remark. In [4] Ward has characterized trees by means of their partial order structure. It is this characterization that we use to prove Theorem 2. We note that the "cut point order" in [4] and the partial order defined in the present paper are the same. We should also note that Ward [6] has investigated the structure of local trees.

THEOREM 2. *An acyclic, arcwise connected continuum X is a tree if and only if for any $e \in X$ the partial order with minimal element e is semi-continuous.*

Proof. If X is a tree, then Ward's results [4, 5] show that the condition holds. Thus, let $e_1 \in X$ and let P_1 be the partial order with minimal element e. Again by Ward [4] we need only show that for any $x \in X$ that the set $xP_1 \setminus x$ is open. Suppose that there exists an $x \in X$ such that $xP_1 \setminus x$ is not open. Let e_2 be an element of $xP_1 \setminus x$ which is not an interior point. Then there is a set $\{x_\gamma \mid \gamma \in A\}$ converging to e_2 such that $x_\gamma \notin [e_1, x] \cup xP_1$ for all $\gamma \in A$. Then with e_2 as a minimal element for P_2 we have $x_\gamma \in xP_2$ for all γ, but $e_2 \notin xP_2$.

This contradicts the assumption that xP_2 is closed. Hence, X must be a tree.

COROLLARY. *An acyclic arcwise connected continuum X is a tree if and only if it is a local tree.*

Proof. If X is a tree, then, by Ward [6], X is a local tree. On the other hand, suppose X is a local tree, and let e be an element of X and P the partial order associated with e. Suppose that $x \in X$ and that $z \in (xP)^*$; first we shall assume that $z \notin Px$. Let U be an open set such that $z \in U$ and $U \cap Px = \Box$. Then let N be a neighborhood of z which is a tree and which is contained in U. Since $z \in (xP)^*$, we have $N \cap (xP) \neq \Box$, and so there is a point $y \in xP$ and an arc $[y, z] \subset N$. But then, since X

is acyclic, $x \in [e, z] = [e, y] \cup [y, z]$, and so $z \in xP$. Finally, a similar argument shows that $z \in (xP)^*$ implies that $z \notin Px$. Thus, $xP = (xP)^*$ for each $x \in X$, and by Theorem 2, X is a tree.

We now examine the example mentioned above.

Example. Let A be the segment from the origin to the point $(2, 0)$ in the plane. For each positive integer n, let I_n be the segment from the origin to the point $(1-1/2^n, 1/2^n)$, and let J_n be the segment from the point $(2, 0)$ to the point $(1+1/2^n, 1/2^n)$. Then define X by

$$X = A \cup \Big(\bigcup_{n=1}^{\infty} I_n\Big) \cup \Big(\bigcup_{n=1}^{\infty} J_n\Big).$$

One can show that X is hereditarily unicoherent and arcwise connected. However, there does not exist a point $e \in X$ such that the partial order with minimal element e is semi-continuous.

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