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CHANGES OF VARIABLE WHICH PRESERVE ALMOST EVERYWHERE APPROXIMATE DIFFERENTIABILITY

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The purpose of this paper is to characterize the *class of changes of variable* (*homeomorphisms*) *which preserve almost everywhere* (*a.e.*) *approximate differentiability*; that is, the class of homeomorphisms H such that, for each approximately differentiable a.e. function F, the composition $F \circ H$ is also approximately differentiable a.e. As a by-product we also find that the same class is the class which preserves a.e. derivability of a real function. For this purpose we need the following definitions:

(i) The *density* of a measurable set E at x_0 is

$$\lim_{|I|\to 0} \frac{|E\cap I|}{|I|}, \quad \text{where } x_0 \in I.$$

(ii) $F'_{\mathrm{ap}}(x_0) = L$ if there exists a measurable set E whose density is 1 at x_0 such that

$$\frac{F(x)-F(x_0)}{x-x_0} \to L,$$

where the limit is taken through $x \neq x_0$, $x \in E$.

(iii) A set E is said to be *open in the density topology* (*D-open*) if each point of E is a point of density (i.e., of density 1).

(iv) A function F is said to be *continuous in the density topology* if the inverse image of D-open sets is D-open.

The following theorem characterizes the class of changes of variable which preserve approximate derivability a.e.

THEOREM. *A homeomorphism H preserves a.e. approximate differentiability if and only if H is continuous a.e. in the density topology and this holds if and only if H^{-1} is absolutely continuous. Furthermore, this result also holds true if the approximate differentiability is replaced by ordinary differentiability.*

The theorem follows from the three lemmas given in the sequel. Note that the measurability of the sets constructed will generally be assumed. The proof of measurability is by standard methods.

LEMMA 1. *Let $H(x)$ be a homeomorphism on an interval $[a, b]$. Then $H(x)$ is continuous in the density topology a.e. if and only if H^{-1} is absolutely continuous.*

Proof. Suppose H^{-1} is not absolutely continuous. Then H^{-1} does not satisfy Lusin's condition (N); i.e., there exists Z in the range of H with $|Z| = 0$ such that $|H^{-1}(Z)| > 0$. Let x be any point of density of $H^{-1}(Z)$ and let $A_x = Z^c \cup \{H(x)\}$ (here Z^c denotes the complement of Z). Since A_x^c is of measure 0, $H(x)$ is a point of density of A_x. However, $H^{-1}(A_x) = H^{-1}(Z^c) \cup \{x\}$ and x is not a point of density of $H^{-1}(A_x)$. Since almost every point of $H^{-1}(Z)$ is a point of density of $H^{-1}(Z)$ (the set $H^{-1}(Z)$ can be chosen to be measurable), H is discontinuous in the density topology at almost every point of $H^{-1}(Z)$.

Suppose H^{-1} is absolutely continuous. Then H does not take any set of positive measure into a set of measure 0. Let A_1 be the set of points x where $H'(x) = 0$, let A_2 be the set of points x where $H'(x)$ is infinite, and let A_3 be the set of points x where $H'(x)$ does not exist. Suppose H is not continuous a.e. in the density topology. From the equalities $|A_2| = |A_3| = 0$ we infer that A_2, A_3 are negligible sets. Since $|H(A_1)| = 0$ [1] and H^{-1} is absolutely continuous, $|A_1| = 0$. Let $B = [a, b] \setminus (A_1 \cup A_2 \cup A_3)$. Without loss of generality, we can assume that H is increasing. Let $B_n = \{x \mid 1/n \leqslant H'(x) \leqslant n\}$. Then $B = \bigcup B_n$. Let D be the set of points where H is discontinuous in the density topology. We show that, for each natural number n and $\eta > 0$, the outer measure of $D \cap B_n$ is less than η. It follows that D is measurable and $|D| = 0$. It suffices to consider n for which $|B_n| > 0$. Since $H'(x)$ is a measurable function, from Lusin's theorem it follows that there is a closed subset B of B_n on which $H'(x)$ is continuous and such that $|B_n \setminus B| < \eta$. Let B' be the set of points of density of B. Then $|B'| = |B|$, and hence $|H(B')| > 0$. Since almost every $y \in H(B')$ is a point of density of $H(B')$, we infer that, except for a set of measure 0, every $x \in B'$ is mapped to a point $y \in H(B')$, where y is a point of density of $H(B')$. Let C be the set B' less the subset of B' of measure 0 whose points do not map into points of density of $H(B')$. All that remains is to show that H is continuous in the density topology at each point of C. Let x_0 be any point in C and let A be a set with $x_0 \in A$ such that $H(A)$ has $H(x_0)$ as a point of density. It remains to show that A has x_0 as a point of density. Since H is a homeomorphism, we have $H(A) \cap H(C) = H(A \cap C)$, and thus $H(A \cap C)$ has $H(x_0)$ as a point of density. Choose $\varepsilon > 0$ so small

(1) See p. 226 in: S. Saks, *Theory of the integral*, New York 1937.

that $H'(x_0) > \varepsilon$. Choose $\delta > 0$ such that if I is an interval with $x_0 \in I$ and $|I| < \delta$, then the maximum value of $H'(x)$ on $I \cap C$ is less than $H'(x_0)+\varepsilon$,

$$|H(I)|/|I| > H'(x_0)-\varepsilon \quad \text{and} \quad |H(I)|/|H(A\cap C\cap I)| < 1+\varepsilon.$$

This can be accomplished because H' is continuous on C, H is differentiable at x_0, and $H(A\cap C)$ has $H(x_0)$ as a point of density. Let I be any interval with $x_0 \in I$ and $|I| < \delta$. Suppose $|A\cap C\cap I| = 0$. Since $H'(x)$ exists at each point of $A\cap C\cap I$, we have $H(A\cap C\cap I) = 0$. But this cannot happen if $H(x_0)$ is to be a point of density of $H(A\cap C)$, so $|A\cap C\cap I| > 0$. Since

$$|H(A\cap C\cap I)| \leqslant |A\cap C\cap I|\big(H'(x_0)+\varepsilon\big),$$

we get

$$\frac{|H(A\cap C\cap I)|}{|A\cap C\cap I|} \leqslant H'(x_0)+\varepsilon. \tag{1}$$

However, from the inequalities

$$\frac{|H(I)|}{|H(A\cap C\cap I)|} \leqslant 1+\varepsilon \tag{2}$$

and

$$\frac{|I|}{|H(I)|} \leqslant \frac{1}{H'(x_0)-\varepsilon} \tag{3}$$

it follows, by combining (1)-(3), that

$$\frac{|I|}{|A\cap C\cap I|} \leqslant (1+\varepsilon)\,\frac{H'(x_0)+\varepsilon}{H'(x_0)-\varepsilon}\,.$$

Thus $A\cap C$ has x_0 as a point of density and H is continuous in the density topology at x_0. Since x_0 was an arbitrary point of C, H is continuous in the density topology at every point of C.

LEMMA 2. *Let $\mathscr{H}$ be the set of homeomorphisms H such that, whenever F is a function which is approximately derivable a.e., $F\circ H$ is also approximately derivable a.e. If H^{-1} is absolutely continuous, then*

$$H \in \mathscr{H} \quad \textit{and} \quad (F\circ H)'_{\mathrm{ap}}(x) = F'_{\mathrm{ap}}\big(H(x)\big)\cdot H'(x) \ \textit{a.e.}$$

Proof. By Lemma 1, if H^{-1} is absolutely continuous, then H is continuous a.e. in the density topology. Let D be the set of points of continuity of H in the density topology and let E be the set of points where $H'(x)$ exists, is finite, and non-zero. Since H^{-1} is absolutely continuous, the set $(H')^{-1}(\{0\})$ has measure 0, its H-image being of measure 0. It follows that almost every point in the domain of H belongs to $E\cap D$. Applying the chain rule, we infer that $F\circ H$ is approximately derivable a.e. For let B be the set of points where $F'_{\mathrm{ap}}(y)$ exists. Then since the relative complement of B in the domain of F has measure 0, $H^{-1}(B)$ is almost all the domain of H. Let $A = H^{-1}(B)\cap D\cap E$, let $x_0 \in A$, and $y_0 = H(x_0)$. Then there is a set G which has y_0 as a point of density on which $F'(y_0)$ exists. $H^{-1}(G)$ has x_0 as a point of density. It follows that

$$(F\circ H)'_{\mathrm{ap}}(x_0) = \lim_{\substack{x\to x_0\\ x\in H^{-1}(G)}} \frac{F(H(x)) - F(H(x_0))}{x - x_0}$$

$$= \lim_{\substack{x\to x_0\\ x\in H^{-1}(G)}} \frac{F(H(x)) - F(H(x_0))}{H(x) - H(x_0)} \lim_{x\to x_0} \frac{H(x) - H(x_0)}{x - x_0}$$

provided both limits exist. But $H(x) \neq H(x_0)$, H' exists at x_0 and F'_{ap} exists at $H(x_0)$ where the limit is taken through points $H(x) = y \in G$. Thus both limits do exist and the lemma is proved.

LEMMA 3. *If H is a homeomorphism and H^{-1} is not absolutely continuous, then there exists a continuous function F whose derivative exists a.e. such that $F\circ H$ is not approximately derivable a.e.* (*That is, the collection of homeomorphisms H such that H^{-1} is absolutely continuous contains the class $\mathscr{H}$.*)

Proof. Suppose H^{-1} is not absolutely continuous. Then there exists a set P such that $|P| > 0$ and $|H(P)| = 0$. Clearly, P can be chosen to be a perfect set. It is well known that there are continuous functions which are nowhere approximately derivable. Let G be such a function. Define G_0 to be G on P and to be linear on intervals contiguous to P. Then G_0 is continuous and is approximately derivable on at most a set of measure 0 of P. Let $F = G_0\circ H^{-1}$. Since G_0 is monotone on intervals contiguous to P, $G_0\circ H^{-1}(x)$ is monotone on intervals contiguous to $H(P)$. Thus $F = G_0\circ H^{-1}$ is differentiable a.e. However, $G_0 = F\circ H$ is not approximately derivable on a set of positive measure (namely, almost all of P). Thus Lemma 3 is proved.

Note. If F is differentiable a.e. and $H \in \mathscr{H}$, then $F\circ H$ is differentiable a.e. For if $H \in \mathscr{H}$, then H^{-1} is absolutely continuous and $A = \{x \mid H'(x)$ exists, is finite, and non-zero$\}$ is almost all of the domain of H. Let $Z = \{y \mid F'(y)$ does not exist$\}$; then $|H^{-1}(Z)| = 0$. Let $x_0 \in A\setminus H^{-1}(Z)$. Then the chain rule applies and we get

$$\lim_{x\to x_0} \frac{F(H(x)) - F(H(x_0))}{x - x_0}$$

$$= \lim_{x\to x_0} \frac{F(H(x)) - F(H(x_0))}{H(x) - H(x_0)} \lim_{x\to x_0} \frac{H(x) - H(x_0)}{x - x_0} = F'(H(x_0))\cdot H'(x_0),$$

where both of the limits in the product exist because $H(x) \neq H(x_0)$, F' exists at $H(x_0)$, and $H'(x_0)$ exists. Thus $\mathscr{H}$ is also the set of changes of variable which preserve the derivative a.e.

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