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DERIVATIVES OF CARTESIAN PRODUCT AND DISPERSED SPACES

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0. Introduction. This paper contains some topological applications of Hessenberg's natural sum of ordinal numbers. Algebraic properties of this operation were studied by Sikorski in [3].

Our Theorem 1 generalizes the known formula for the derivative, i.e. the set of limit points, of a cartesian product of sets in topological spaces. Theorem 2 gives a topological definition of the natural sum and some applications to dispersed spaces. Finally, we give conditions under which the derivative of a set is closed and other related facts as well as proofs of the theorems.

It seems that it is the first time that Hessenberg's sum found an application apparently distant from its definition.

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1. Main results. $A^{(1)}$ denotes the set of all limit points of a set A in a topological space X and it is called the *derivative* of A. We put $A^{(0)} = \overline{A}$, where $\overline{A}$ is the closure of A, and

$$A^{(\gamma)} = \bigcap_{\alpha<\gamma} (A^{(\alpha)})^{(1)}$$

for $\gamma > 0$. If $A^{(1)}$ is a closed set, then $A^{(\alpha)}$ is also closed for all $\alpha > 0$ (see (2) in section 3), and if $0 < \alpha \leqslant \beta$, then $A^{(\beta)} \subseteq A^{(\alpha)}$.

It is well known that

$$(A\times B)^{(1)} = A^{(1)}\times \overline{B} \cup \overline{A}\times B^{(1)} = A^{(1)}\times B^{(0)} \cup A^{(0)}\times B^{(1)},$$

where $\times$ denotes the cartesian product operation. From this formula it follows by induction that

$$(*) \qquad (A\times B)^{(n)} = \bigcup_{i+j=n} A^{(i)}\times B^{(j)} \quad \text{for all } n\in\omega_0.$$

$\alpha\oplus\beta$ denotes Hessenberg's natural sum of ordinals α, β (see [2] or [3]) which can be defined as follows:

Let $\alpha = \alpha_1 + \ldots + \alpha_m$, $\beta = \beta_1 + \ldots + \beta_n$ be the expansions of any given ordinals α, β into prime components (see [2], p. 282), where $\alpha_1 \geqslant \ldots \geqslant \alpha_m$ and $\beta_1 \geqslant \ldots \geqslant \beta_n$. As it is well known, these expansions are unique. Let $\gamma_1, \ldots, \gamma_{m+n}$ be the sequence formed from the sequences $\alpha_1, \ldots, \alpha_m$ and $\beta_1, \ldots, \beta_n$ in such a way that $\gamma_1 \geqslant \ldots \geqslant \gamma_{m+n}$. Now we define $\alpha \oplus \beta = \gamma_1 + \ldots + \gamma_{m+n}$.

Obviously we have $\alpha \oplus 0 = \alpha$, $\alpha \oplus \beta = \beta \oplus \alpha$, $\alpha \oplus (\beta + 1) = (\alpha \oplus \beta) + 1$, $\alpha_1 < \alpha_2 \Leftrightarrow (\alpha_1 \oplus \beta < \alpha_2 \oplus \beta)$.

The following result generalizes formula (*).

THEOREM 1. *If the sets $A^{(1)}$ and $B^{(1)}$ are closed, then for each ordinal α*

$$(A \times B)^{(\alpha)} = \bigcup_{\mu \oplus \nu = \alpha} A^{(\mu)} \times B^{(\nu)}.$$

Let us note that the conditions of Theorem 1 are always satisfied in T_1-spaces. This known fact follows also from (4) in section 3.

Remark 1. For each ordinal α the set $\{\langle \mu, \nu \rangle : \mu \oplus \nu = \alpha\}$ is finite, because the expansions into prime components contain only finitely many summands.

If γ is any prime component and the sets $A^{(1)}$, $B^{(1)}$ are closed, then

$$(A \times B)^{(\gamma)} = A^{(\gamma)} \times B^{(0)} \cup A^{(0)} \times B^{(\gamma)} = A^{(\gamma)} \times \bar{B} \cup \bar{A} \times B^{(\gamma)}.$$

X is called a *dispersed space* if there exists an ordinal α such that $X^{(\alpha)} = 0$. The first ordinal α for which $X^{(\alpha)} = 0$ holds will be denoted by $\xi(X)$ and will be called the *rank of dispersion* of the space X. The *rank of a point* x in a dispersed space X is defined by the following formula:

$$\varrho(x) = \alpha \text{ if and only if } x \in X^{(\alpha)} - X^{(\alpha+1)}.$$

THEOREM 2. *If X and Y are dispersed spaces and $\langle x, y \rangle \in X \times Y$, then $\varrho(\langle x, y \rangle) = \varrho(x) \oplus \varrho(y)$.*

As A. Mostowski remarked, this theorem gives a topological definition of natural sum of ordinals. It is sufficient to take the class of all ordinals with interval topology (open intervals and the set $\{0\}$ form a basis for open sets), and to see that $\varrho(\omega^\alpha) = \alpha$ for each ordinal α. Then $\alpha \oplus \beta = \varrho(\langle \omega^\alpha, \omega^\beta \rangle)$.

We put $Z_{(\alpha)} = Z^{(\alpha)} - Z^{(\alpha+1)}$ for any dispersed space Z. Then we have some simple corollaries from Theorem 2, the second of which is a solution of a problem proposed by B. Węglorz.

COROLLARY 1. *If X and Y are dispersed spaces, then* $(X \times Y)_{(\alpha)} = \bigcup \{X_{(\mu)} \times Y_{(\nu)} : \mu \oplus \nu = \alpha\}$.

COROLLARY 2. *If X and Y are dispersed spaces, then* $\xi(X \times Y) = \bigcup \{\alpha \oplus \beta + 1 : \alpha < \xi(X) \wedge \beta < \xi(Y)\}$ (1).

(1) As usual each ordinal is identified with the set of smaller ordinals. Hence both operations $\bigcup$ and sup coincide on sets of ordinals.

Really, it is easy to see that

$$\begin{aligned}\xi(X\times Y) &= \bigcup\{\varrho(\langle x, y\rangle)+1\colon x\in X\wedge y\in Y\}\\ &= \bigcup\{\varrho(x)\oplus\varrho(y)+1\colon x\in X\wedge y\in Y\}\\ &= \bigcup\{\alpha\oplus\beta+1\colon \alpha<\xi(X)\wedge\beta<\xi(Y)\}.\end{aligned}$$

Remark 2. Our formulas for $(A\times B)^{(\alpha)}$, $(X\times Y)_{(\alpha)}$, $\varrho(\langle x, y\rangle)$ and $\xi(X\times Y)$ can be generalized to any finite cartesian product, i.e.

$$(A_1\times\ldots\times A_n)^{(\alpha)} = \bigcup\{A_1^{(\alpha_1)}\times\ldots\times A_n^{(\alpha_n)}\colon \alpha_1\oplus\ldots\oplus\alpha_n = \alpha\},$$

if $A_1^{(1)}, \ldots, A_n^{(1)}$ are closed;

$$(X_1\times\ldots\times X_n)_{(\alpha)} = \bigcup\{(X_1)_{(\alpha_1)}\times\ldots\times(X_n)_{(\alpha_n)}\colon \alpha_1\oplus\ldots\oplus\alpha_n = \alpha\},$$

$$\xi(X_1\times\ldots\times X_n) = \bigcup\{\alpha_1\oplus\ldots\oplus\alpha_n+1\colon \alpha_1<\xi(X_1)\wedge\ldots\wedge\alpha_n<\xi(X_n)\},$$

$$\varrho(\langle x_1, \ldots x_n\rangle) = \varrho(x_1)\oplus\ldots\oplus\varrho(x_n)$$

for $\langle x_1, \ldots, x_n\rangle\in X_1\times\ldots\times X_n$, if $X_1, \ldots, X_n$ are dispersed spaces.

It is not interesting to extend the above formulas to cartesian products of infinitely many dispersed spaces because if infinitely many spaces among them have more than one point, then the product space has no isolated points and hence all the derivatives are equal.

Remark 3. Theorem 2 and Corollaries 1 and 2 are valid also in all T_1-spaces, if $\xi(X)$ and $\varrho(x)$ are extended in the following way:

$\xi(X)$ is the first ordinal α such that $X^{(\alpha)} = \hat{X}$, where $\hat{X}$ is the *kernel* of the space X, i.e.

$$\hat{X} = \bigcap_{\alpha} X^{(\alpha)}.$$

(A T_1-space X is dispersed if and only if $\hat{X} = 0$; see [1], p. 5.)

$\varrho(x)$ is defined for $x\in X-\hat{X}$ as above and $\varrho(x) = \infty$ for $x\in\hat{X}$.

If we put $\alpha\oplus\infty = \infty\oplus\alpha = \infty$ and $\infty\oplus\infty = \infty$, then we get the desired results.

2. Conditions for $A^{(1)}$ to be closed. We have found two kinds of conditions.

(I) Conditions imposed on $A^{(1)}$ only, which can be presented in the following diagram:

$$\begin{tikzcd}
 & & A \subseteq A^{(1)} \arrow[dr, Rightarrow] & \\
\bigwedge\limits_{x\in A} x\notin\{x\}^{(2)} \arrow[r, Leftrightarrow] & \bigwedge\limits_{x\in A}\overline{\{x\}^{(1)}} = \{x\}^{(1)} \arrow[rr, Rightarrow] & & \overline{A^{(1)}} = A^{(1)} \\
 & & A = \overline{A} \arrow[ur, Rightarrow] &
\end{tikzcd}$$

(II) Conditions concerning all $Y^{(1)}$, where $Y \subseteq X$, which can be presented in the following diagram:

$$\begin{array}{r} X \text{ is } T_1 \searrow \\ X \text{ is dispersed} \nearrow \end{array} \bigwedge_{x\in X} \overline{\{x\}^{(1)}} = \{x\}^{(1)} \Leftrightarrow \bigwedge_{x\in X} x \notin \{x\}^{(2)} \Leftrightarrow \bigwedge_{Y\subseteq X} \overline{Y^{(1)}} = Y^{(1)}.$$

The next diagram presents the consequences of the above conditions:

$$\bigwedge_{Y\subseteq X} \overline{Y^{(1)}} = Y^{(1)} \Rightarrow X \text{ is } T_0 \Rightarrow \bigwedge_{Y\subseteq X} \overline{Y^{(2)}} = Y^{(2)}.$$

No implication in these diagrams can be conversed.

3. Proofs. To prove Theorem 1 we need the following two lemmas:

LEMMA 1. *If γ is a limit ordinal and $\gamma = \mu\oplus\nu$, then there exist sequences $\{\mu_a\}_{a<\gamma}$ and $\{\nu_a\}_{a<\gamma}$ such that* (i) *$\mu_a\oplus\nu_a = a$ for each $a<\gamma$,* (ii) *$\bigcup_{a<\gamma}\mu_a = \mu$ and* (iii) *$\bigcup_{a<\gamma}\nu_a = \nu$.*

Proof. Let γ be any limit ordinal and let $\gamma = \mu\oplus\nu$. Let us split the expansion of $\gamma = \gamma_1+\ldots+\gamma_n$ (into prime components) into segments $\beta_1 = \gamma_1+\ldots+\gamma_{k_1}$, $\beta_2 = \gamma_{k_1+1}+\ldots+\gamma_{k_2}, \ldots, \beta_r = \gamma_{k_{r-1}+1}+\ldots+\gamma_n$ in such a way that $\beta_1+\beta_3+\ldots = \mu$ and $\beta_2+\beta_4+\ldots = \nu$ or conversely $\beta_1+\beta_3+ +\ldots = \nu$ and $\beta_2+\beta_4+\ldots = \mu$. Without any lost of generality we may assume that the first possibility occurs.

Now we can define sequences $\{\mu_a\}_{a<\gamma}$, $\{\nu_a\}_{a<\gamma}$ as follows:

for $a\leqslant\beta_1$ we have $\mu_a = a$ and $\nu_a = 0$;

for $\beta_1 < a \leqslant \beta_1+\beta_2$ we have $\mu_a = \beta_1$ and ν_a satisfies the equation $\beta_1\oplus\nu_a = a$;

for $\beta_1+\beta_2 < a \leqslant \beta_1+\beta_2+\beta_3$ we have $\nu_a = \beta_2$ and $\mu_a = \beta_1+\xi_a$, where ξ_a satisfies the equation $(\beta_1+\beta_2)\oplus\xi_a = a$;

for $\beta_1+\beta_2+\beta_3 < a \leqslant \beta_1+\beta_2+\beta_3+\beta_4$ we have $\mu_a = \beta_1+\beta_3$ and $\nu_a = \beta_2+\xi_a$, where ξ_a satisfies the equation $(\beta_1+\beta_2+\beta_3)\oplus\xi_a = a$; and so on.

This definition gives $\{\mu_a\}_{a<\gamma}$, $\{\nu_a\}_{a<\gamma}$ having desired properties if the equations of the form

$$(\beta_1+\ldots+\beta_{s-1})\oplus\xi_a = a \quad \text{for} \quad \beta_1+\ldots+\beta_{s-1} < a \leqslant \beta_1+\ldots+\beta_s,$$

where $s\leqslant r$, always have a solution.

Let $\beta_1+\ldots+\beta_{s-1} < a \leqslant \beta_1+\ldots+\beta_s$. Then there is a maximal k such that $\gamma_1+\ldots+\gamma_k < a$. But then $a \leqslant \gamma_1+\ldots+\gamma_k+\gamma_{k+1}$ and there exists η_a such that $\eta_a \leqslant \gamma_{k+1}$ and $\gamma_1+\ldots+\gamma_k+\eta_a = a$. Let $\sigma_1+\ldots+\sigma_t$ be the normal expansion of η_a. To prove $(\gamma_1+\ldots+\gamma_k)\oplus\eta_a = \gamma_1+\ldots+\gamma_k+$

$+\eta_a$ it is sufficient to show that $\sigma_i \leqslant \gamma_k$ for $i = 1, 2, \ldots, t$. If $\sigma_i > \gamma_k$ for some i, then we would have $\gamma_{k+1} \geqslant \eta_a \geqslant \sigma_i > \gamma_k$, which is not true, because $\gamma_{k+1} \leqslant \gamma_k$.

Let p be a natural number such that $\beta_1 + \ldots + \beta_{s-1} = \gamma_1 + \gamma_2 + \ldots + \gamma_p$ and let $\xi_a = \gamma_{p+1} + \ldots + \gamma_k + \sigma_1 + \sigma_2 + \ldots + \sigma_t$. Then

$$(\beta_1 + \ldots + \beta_{s-1}) \oplus \xi_a = (\gamma_1 + \ldots + \gamma_p) \oplus (\gamma_{p+1} + \ldots + \gamma_k + \sigma_1 + \ldots + \sigma_t)$$
$$= \gamma_1 + \gamma_2 + \ldots + \gamma_k + \eta_a = a,$$

hence ξ_a is the desired solution.

From the definition of $\{\mu_a\}_{a<\gamma}$ and $\{\nu_a\}_{a<\gamma}$ we see that they have properties (i), (ii) and (iii), q. e. d.

LEMMA 2. *If* $\mu \oplus \nu > a$, *then there are* μ_a *and* ν_a *such that* $\mu_a \oplus \nu_a = a$, $\mu_a \leqslant \mu$ *and* $\nu_a \leqslant \nu$.

Proof. We prove this by induction with respect to $\gamma = \mu \oplus \nu > a$.

For $\gamma = 0$ this is trivial.

Suppose that for some γ the conclusion is true. If $\gamma + 1 = \mu \oplus \nu > a$, then either μ or ν or both are of the form $\beta + 1$, for instance $\mu = \beta + 1$. But then $\gamma + 1 = (\beta + 1) \oplus \nu = (\beta \oplus \nu) + 1$ and $\gamma = \beta \oplus \nu \geqslant a$. By inductive supposition there are β_a and ν_a having the desired properties. Putting $\mu_a = \beta_a$ we get $\mu_a = \beta_a \leqslant \beta < \beta + 1 = \mu$, $\nu_a \leqslant \nu$ and $\mu_a \oplus \nu_a = \beta_a \oplus \nu_a = a$.

For a limit ordinal γ the assertion of Lemma 2 follows from Lemma 1, q. e. d.

Proof of Theorem 1. We prove this by induction with respect to a.

For $a = 0$ the theorem holds, because $(A \times B)^{(0)} = \overline{A \times B} = \overline{A} \times \overline{B} = A^{(0)} \times B^{(0)}$.

Suppose that for some a the theorem holds. Then

$$(A \times B)^{(a+1)} = \Big(\bigcup_{\mu \oplus \nu = a} A^{(\mu)} \times B^{(\nu)}\Big)^{(1)} = \bigcup_{\mu \oplus \nu = a} (A^{(\mu)} \times B^{(\nu)})^{(1)}$$

$$= \bigcup_{\mu \oplus \nu = a} (A^{(\mu+1)} \times \overline{B^{(\nu)}} \cup \overline{A^{(\mu)}} \times B^{(\nu+1)}) = \bigcup_{\mu \oplus \nu = a} (A^{(\mu+1)} \times B^{(\nu)} \cup A^{(\mu)} \times B^{(\nu+1)})$$

$$= \bigcup_{\mu \oplus \nu = a} (A^{(\mu+1)} \times B^{(\nu)} \cup \bigcup_{\mu \oplus \nu = a} A^{(\mu)} \times B^{(\nu+1)}$$

$$= \bigcup_{(\mu+1) \oplus \nu = a+1} (A^{(\mu+1)} \times B^{(\nu)} \cup A^{(\nu)} \times B^{(\mu+1)} \cup \bigcup_{\mu \oplus (\nu+1) = a+1} (A^{(\mu)} \times B^{(\nu+1)} \cup A^{\nu+1)} \times B^{(\mu)})$$

$$= \bigcup_{\mu \oplus \nu = a+1} (A^{(\mu)} \times B^{(\nu)} \cup A^{(\nu)} \times B^{(\mu)}) \cup \bigcup_{\mu \oplus \nu = a+1} (A^{(\mu)} \times B^{(\nu)} \cup A^{(\nu)} \times B^{(\mu)})$$

$$= \bigcup_{\mu \oplus \nu = a+1} A^{(\mu)} \times B^{(\nu)}.$$

Let γ be any limit ordinal number. Suppose that for each $\alpha < \gamma$ the theorem holds. Then

$$(A \times B)^{(\gamma)} = \bigcap_{\alpha<\gamma} (A \times B)^{(\alpha)} = \bigcap_{\alpha<\gamma} \bigcup_{\mu\oplus\nu=\alpha} A^{(\mu)} \times B^{(\nu)}$$
$$= \bigcup \{\bigcap_{\alpha<\gamma} A^{(\mu_\alpha)} \times B^{(\nu_\alpha)}\} = \bigcup \{A^{(\overline{\mu_\gamma})} \times B^{(\overline{\nu_\gamma})}\}$$

where the unions are taken over all sequences $\{\mu_\alpha\}_{\alpha<\gamma}$ and $\{\nu_\alpha\}_{\alpha<\gamma}$ such that $\mu_\alpha \oplus \nu_\alpha = \alpha$ for each $\alpha < \gamma$, $\overline{\mu_\gamma} = \bigcup_{\alpha<\gamma} \mu_\alpha$, $\overline{\nu_\gamma} = \bigcup_{\alpha<\gamma} \nu_\alpha$.

From Lemma 1 it follows that

$$\bigcup_{\mu\oplus\nu=\gamma} A^{(\mu)} \times B^{(\nu)}$$
$$\subseteq \bigcup \{A^{(\overline{\mu_\gamma})} \times B^{(\overline{\nu_\gamma})}\colon \{\mu_\alpha\}_{\alpha<\gamma}, \{\nu_\alpha\}_{\alpha<\gamma}, \mu_\alpha \oplus \nu_\alpha = \alpha \text{ for each } \alpha < \gamma\}.$$

In order to prove the converse inclusion let us remark that $\alpha = \mu_\alpha \oplus \nu_\alpha \leqslant \overline{\mu_\gamma} \oplus \overline{\nu_\gamma}$, and hence

$$\gamma = \bigcup_{\alpha<\gamma} \alpha \leqslant \overline{\mu_\gamma} \oplus \overline{\nu_\gamma}.$$

For given sequences $\{\mu_\alpha\}_{\alpha<\gamma}$, $\{\nu_\alpha\}_{\alpha<\gamma}$ satisfying $\mu_\alpha \oplus \nu_\alpha = \alpha$ for each $\alpha < \gamma$, by Lemma 2 there are μ, ν such that $\mu \leqslant \overline{\mu_\gamma}$, $\nu \leqslant \overline{\nu_\gamma}$ and $\mu \oplus \nu = \gamma$. By supposition the sets $A^{(1)}$ and $B^{(1)}$ are closed and hence $A^{(\overline{\mu_\gamma})} \subseteq A^{(\mu)}$ and $B^{(\overline{\nu_\gamma})} \subseteq B^{(\nu)}$. This means that

$$A^{(\overline{\mu_\gamma})} \times B^{(\overline{\nu_\gamma})} \subseteq \bigcup_{\mu\oplus\nu=\gamma} A^{(\mu)} \times B^{(\nu)}$$

for any sequences $\{\mu_\alpha\}_{\alpha<\gamma}$ and $\{\nu_\alpha\}_{\alpha<\gamma}$ such that $\mu_\alpha \oplus \nu_\alpha = \alpha$ for each $\alpha < \gamma$.

This completes our inductive proof of Theorem 1.

Proof of Theorem 2. Let $\langle x, y\rangle \in X \times Y$, where X, Y are dispersed spaces. Let $\alpha = \varrho(\langle x, y\rangle)$. Then

$$\langle x, y\rangle \in (X \times Y)^{(\alpha)} - (X \times Y)^{(\alpha+1)} \subseteq \bigcup_{\mu\oplus\nu=\alpha} X^{(\mu)} \times Y^{(\nu)},$$

and hence there are μ and ν such that $x \in X^{(\mu)}$, $y \in Y^{(\nu)}$ and $\mu \oplus \nu = \alpha$. Hence $\mu \leqslant \varrho(x)$ and $\nu \leqslant \varrho(y)$. If we had $\mu < \varrho(x)$, then $\beta = \varrho(x) \oplus \varrho(y) > \mu \oplus \nu = \alpha$, because the function $\oplus$ is strictly increasing. Then $\langle x, y\rangle \in X^{(\varrho(x))} \times Y^{(\varrho(y))} \subseteq (X \times Y)^{(\beta)}$, but this is a contradiction, because $\beta > \alpha$ implies

$$[(X \times Y)^{(\alpha)} - (X \times Y)^{(\alpha+1)}] \cap (X \times Y)^{(\beta)} = 0.$$

Hence the inequality $\mu \geqslant \varrho(x)$ holds, which gives $\mu = \varrho(x)$. Similarly $\nu = \varrho(y)$, and $\varrho(x) \oplus \varrho(y) = \mu \oplus \nu = \alpha = \varrho(\langle x, y\rangle)$, q. e. d.

In order to prove the implications in diagrams of section 2 it is useful to have the following

LEMMA 3. *If* $x \in A^{(2)} - A^{(1)}$, *then* $x \in \{x\}^{(2)}$.

Proof. Suppose to the contrary that there exists an x such that $x \in A^{(2)} - A^{(1)}$ and $x \notin \{x\}^{(2)}$. $x \notin A^{(1)}$ means that there exists a neighbourhood U_x of x such that $U_x \cap A - \{x\} = 0$. Hence $U_x \cap \overline{A - \{x\}} = 0$ and also $U_x \cap (A - \{x\})^{(1)} = 0$. Similarly $x \notin \{x\}^{(2)}$ means that there exists a neighbourhood V_x of x such that $V_x \cap \{x\}^{(1)} - \{x\} = 0$. But $x \notin \{x\}^{(1)}$, so we can write $V_x \cap \{x\}^{(1)} = 0$. Let $W_x = U_x \cap V_x$. Then

$$\begin{aligned} W_x \cap (A \cup \{x\})^{(1)} &= W_x \cap (A - \{x\} \cup \{x\})^{(1)} \\ &= \big(W_x \cap (A - \{x\})^{(1)}\big) \cup (W_x \cap \{x\}^{(1)}) = 0, \end{aligned}$$

and hence $x \notin \overline{(A \cup \{x\})^{(1)}} \supseteq \overline{A^{(1)}} \supseteq A^{(2)}$, which contradicts our supposition $x \in A^{(2)}$, q. e. d.

Now we prove the implications of the diagrams (I) and (II).

(1) *If* $A \subseteq A^{(1)}$, *then* $\overline{A^{(1)}} = A^{(1)}$.

Proof. If $A \subseteq A^{(1)}$, then $\overline{A^{(1)}} \subseteq \overline{\overline{A}} = \overline{A} = A \cup A^{(1)} = A^{(1)} \subseteq \overline{A^{(1)}}$, q. e. d.

(2) *If* $\overline{A} = A$, *then* $\overline{A^{(1)}} = A^{(1)}$.

Proof. $A^{(2)} \subseteq (\overline{A})^{(1)} = A^{(1)}$, since $\overline{A} = A$. Hence $\overline{A^{(1)}} = A^{(1)} \cup A^{(2)} = A^{(1)}$, q. e. d.

(3) $\overline{\{x\}^{(1)}} = \{x\}^{(1)}$ *if and only if* $x \notin \{x\}^{(2)}$.

Proof. If $\{x\}^{(1)}$ is closed, then $\{x\}^{(2)} \subseteq \{x\}^{(1)}$. But then $x \notin \{x\}^{(2)}$, since $x \notin \{x\}^{(1)}$. Conversely, if $x \notin \{x\}^{(2)}$, then $\{x\}^{(2)} \subseteq \overline{\{x\}} - \{x\} = \{x\}^{(1)}$; so $\{x\}^{(1)}$ is closed, q. e. d.

(4) *If* $\bigwedge_{x \in A} \overline{\{x\}^{(1)}} = \{x\}^{(1)}$, *then* $\overline{A^{(1)}} = A^{(1)}$.

Proof. If $A^{(1)}$ is not closed, then $A^{(2)} \nsubseteq A^{(1)}$ and therefore $A^{(2)} - A^{(1)} \neq 0$. Let $x \in A^{(2)} - A^{(1)}$; then, by Lemma 3, $x \in \{x\}^{(2)}$ and, by (3), $\{x\}^{(1)}$ is not closed. Let us remark that $A^{(2)} - A^{(1)} \subseteq A$; this means that $x \in A$, q. e. d.

If X is a T_1-space, then $\{x\}^{(1)} = 0$ for each $x \in X$, because $\overline{\{x\}} = \{x\}$. Hence $\{x\}^{(1)}$ is closed for each $x \in X$. Equivalence of $\bigwedge_{x \in A} x \notin \{x\}^{(2)}$ and $\bigwedge_{A \subseteq X} \overline{A^{(1)}} = A^{(1)}$ follows from (4) and (3). Equivalence of $\bigwedge_{x \in X} \overline{\{x\}^{(1)}} = \{x\}^{(1)}$ and $\bigwedge_{x \in X} x \notin \{x\}^{(2)}$ is asserted in (3).

(5) *If* X *is a dispersed space, then* $x \notin \{x\}^{(2)}$ *for each* $x \in X$.

Proof. If there is an $x \in X$ such that $x \in \{x\}^{(2)}$, then $x \in X^{(\varrho(x))}$ and also $x \in \{x\}^{(2)} \subseteq X^{(\varrho(x)+2)}$. The last inclusion is impossible, because X, by supposition, is a dispersed space, q. e. d.

(6) *If* $\{x\}^{(1)}$ *is closed for each* $x \in X$, *then* X *is a* T_0*-space.*

Proof. Let x, y be two distinct points of X. Two cases are possible:

(a) $y \in \overline{\{x\}}$, i.e. $y \in \{x\}^{(1)}$.

Then $U_x = X - \{x\}^{(1)}$ is an open neighbourhood of x such that $y \notin U_x$.

(b) $y \notin \overline{\{x\}}$.

Then $U_y = X - \overline{\{x\}}$ is an open neighbourhood of y such that $x \notin U_y$. Hence the space X is a T_0-space, q. e. d.

(7) *If X is a T_0-space, then $A^{(2)}$ is closed for each $A \subseteq X$.*

Proof. It is sufficient to show that if $x \notin A^{(2)}$, then $x \notin \overline{A^{(2)}}$. Let $x \notin A^{(2)}$. Then there is a neighbourhood U_x of x such that $U_x \cap A^{(1)} - \{x\} = 0$. Two cases may occur:

(a) $U_x \cap A^{(1)} = 0$. Then $U_x \cap \overline{A^{(1)}} = 0$ and hence $x \notin \overline{A^{(1)}} \supseteq \overline{A^{(2)}}$.

(b) $U_x \cap A^{(1)} = \{x\}$, i.e. $x \in A^{(1)}$.

Two cases may occur:

(b_1) $x \notin \{x\}^{(2)}$.

Then, by (3), $\{x\}^{(1)}$ is closed and also $x \notin \{x\}^{(1)}$. $V_x = X - \{x\}^{(1)}$ is an open neighbourhood of x and $V_x \cap \{x\}^{(1)} = 0$. Let $W_x = U_x \cap V_x$. Then $W_x \cap A^{(1)} - \{x\} = 0$, $W_x \cap \overline{A^{(1)} - \{x\}} = 0$, $W_x \cap (A^{(1)} - \{x\})^{(1)} = 0$ and also $W_x \cap \{x\}^{(1)} = 0$. Hence

$$W_x \cap A^{(2)} = W_x \cap (A^{(1)} - \{x\} \cup \{x\})^{(1)} = W_x \cap (A^{(1)} - \{x\})^{(1)} \cup W_x \cap \{x\}^{(1)} = 0,$$

so $x \notin \overline{A^{(2)}}$.

(b_2) $x \in \{x\}^{(2)}$.

Now we prove that $\{x\}^{(1)} \subseteq A^{(1)}$. If $y \in \{x\}^{(1)}$, then $x \in U_y - \{y\}$ for each neighbourhood U_y of y. But then $U_y - \{x\} \cap A \neq 0$, because $x \in A^{(1)}$ and each neighbourhood of y is also a neighbourhood of x. Since X is a T_0-space, there is a neighbourhood V_x of x such that $y \notin V_x$. Now we have $U_y - \{x\} \cap V_x \cap A \neq 0$, i.e. $U_y - \{x\} \cap V_x - \{y\} \cap A \neq 0$, $U_y - \{y\} \cap A \neq 0$ for each U_y. Hence $y \in A^{(1)}$.

But if $\{x\}^{(1)} \subseteq A^{(1)}$, then $\{x\}^{(2)} \subseteq A^{(2)}$ and $x \in \{x\}^{(2)} \subseteq A^{(2)}$, so $x \in A^{(2)}$. This is a contradiction. Hence this case is impossible. Thus (7) is proved, q. e. d.

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