COLLOQUIUM MATHEMATICUM

VOL. XXV 1972 FASC. 1

CERTAIN SUBDIRECT SUMS OF FINITE PRIME FIELDS

BY

FERENC SZÁSZ (BUDAPEST)

The fundamental notions used in this paper can be found in Jacobson [6], Kaplansky [8] and McCoy [12]. All rings considered here will be associative. For arbitrary subsets B and C of a ring A the product BC will mean the additive subgroup generated by all elements bc with $b \in B$ and $c \in C$. The ring of rational integers will be denoted by I. For any element a of the ring A, Ia is the cyclic subgroup generated by a. Following Kandô [7], a ring A is called *strongly regular* if $a \in a^2 A$ for any $a \in A$. Some characterizations of strongly regular rings have been given by Forsythe and McCoy [4], Kovács [9], Lajos – author [11] and author [16] (cf. also [17]). In part II of [11] it is shown that a ring is strongly regular if and only if its multiplicative semigroup is a semilattice of groups. Semigroups which are semilattices of groups (for their definition see Clifford [2]) were characterized also by Lajos [10].

The Boolean rings in which $a^2 = a$ holds for any element a of the ring as well as the discrete direct sums of division rings are important instances of strongly regular rings. Any strongly regular rings is a subdirect sum of division rings [4]. On the other hand, the ring I is a subdirect sum of the prime fields $I/(p)$, where p runs over the set of all prime numbers, but I is not strongly regular. We shall call ring A a *restricted Boolean ring* (or an *MPR-ring*, respectively) if $a^2 = a$ and $ab = ba = a$, or b, or 0 for any $a, b \in A$ (or if A satisfies the minimum condition on principal right ideals of A, respectively; MPR-ring was in German denoted as "MHR-Ring", cf. [15]). As was shown by Gerčikov [5], a ring is a direct discrete sum of division rings if and only if it is an MPR-ring without non-zero nilpotent elements. Furthermore, by Satz 2.5 of part II (page 422) of [15], an MPR-ring A has no non-zero nilpotent elements if and only if any right ideal R, contained in a principal right ideal $(a)_r = Ia + aA$ of A, contains a right unity element of R. Therefore Satz 2.5 of [15] yields also a characterization of discrete direct sums of division rings.

The aim of this paper is to characterize certain strongly regular subdirect sums of finite prime fields.

THEOREM 1. *For a ring A the following two conditions are equivalent:*

(I) *any additive subgroup S of A is multiplicatively idempotent.*

(II) *A is a direct sum of its ideals A_2 and A_p, i.e. $A = A_2 \oplus \sum_p A_p$, where A_2 is a restricted Boolean ring, p runs over the set of all different odd primes, and either $A_p \cong I/(p)$ or $A_p = 0$.*

COROLLARY 2. *Any ring with condition (I) is a subdirect sum of finite prime fields.*

COROLLARY 3. *A ring A without non-zero elements of odd additive order satisfies condition* (I) *if and only if it is a restricted Boolean ring.*

COROLLARY 4. *A ring A without non-zero elements of even additive order satisfies condition* (I) *if and only if it is a torsion ring such that any non-zero p-component A_p of A is isomorphic to $I/(p)$ (where $p \neq 2$).*

Proof of Theorem 1. Assume that A is a ring satisfying condition (I). Since the cyclic group Ia is idempotent for any $a \in A$, there exists a number $m \in I$ such that $a = ma^2$. It can be noted that $e = ma$ is by

$$e^2 = m^2a^2 = m \cdot ma^2 = ma = e$$

idempotent. Furthermore, by

$$a = m^2a^3 = a^2 \cdot m^2 a \in a^2 A \quad \text{for any } a \in A,$$

A is strongly regular and so it has no non-zero nilpotent elements.

We shall show that any element of A has a square free additive order, that is, the additive group A^+ is elementary (cf. Kaplansky [8]). Namely, if $a = ma^2 \neq 0$, then $a^2 \neq 0$. Let p be a prime number which does not divide the number m. Then by condition (I) there exists a number $n \in I$ such that $pa = n(pa)^2$, whence

$$(m-pn)pa^2 = pma^2 - np^2a = pa - pa = 0.$$

This means, by $a^2 \neq 0$, $p \neq 0$ and $m \neq pn$, that A^+ is not torsion free. If T is the maximal torsion ideal of A, then the torsion free ring A/T also satisfies condition (I); consequently, we have $A/T = 0$ and $T = A$. Let now, for an arbitrary prime number p, A_p be a p-component of A. Then $A_p^2 = A_p$ and $(pA_p)^2 = pA_p$ imply $pA_p = p^2A_p$. Hence pA_p^+ is a divisible abelian group, which is, by $T = A$, a direct sum of Prüferian quasicyclic groups $C(p^\infty)$. Obviously, any $C(p^\infty)$ admits only trivial multiplication upon itself (i.e., $xy = 0$ for any $x, y \in C(p^\infty)$), contrary to condition (I). Consequently, for any p, $pA_p = 0$ (cf. I. Kaplansky [8]).

Since A has no non-zero nilpotent elements, any idempotent belongs, according to a result of Forsythe and McCoy [4], to the centre C of A.

Therefore

$$ma = e \in C \quad \text{for } a = ma^2 \in A.$$

Consequently, $a^2 = a$ and $b^2 = b$ imply $ab = ba$.

We shall show that if p is an odd prime number, then $A_p \neq 0$ implies $A_p \cong I/(p)$. For assume the existence of non-zero elements a and b with $Ia \cap Ib = 0$. Then a and b can be chosen, by condition (I), such that $a^2 = a$ and $b^2 = b$. Since $S = Ia + Ib = S^2$ is a subring, we have $ab = ba = ka + lb$ with $k, l \in I$. Now $Ia \cap Ib = 0$, $a^2 b = ab$ and $ab^2 = ab$ yield

$$k^2 \equiv k, \quad l^2 \equiv l \quad \text{and} \quad kl \equiv 0 \pmod{p};$$

consequently, $ab = 0$, or a, or b.

If $ab = ba = 0$, then there exists a number $s \in I$ such that

$$a + 2b = s(a + 2b)^2 = sa + 4sb,$$

whence, by $Ia \cap Ib = 0$,

$$s \equiv 1, \quad 4s \equiv 2 \quad \text{and} \quad 4 \equiv 2 \pmod{p},$$

and so we get $p = 2$ in a contradiction with the assumption $p \neq 2$. Similarly, if $ab = ba = a$, then there exists a number $t \in I$ such that

$$a - 2b = t(a - 2b)^2 = t(-3a + 4b)$$

whence, by $Ia \cap Ib = 0$,

$$3t \equiv -1, \quad 4t \equiv -2, \quad t \equiv -1 \quad \text{and} \quad -3 \equiv -1 \pmod{p},$$

and so we get the same contradiction $p = 2$ with the assumption $p \neq 2$. The case $Ia \cap Ib = 0$, $ab = ba = b$ is similarly impossible.

Therefore we have $A_p \cong I/(p)$ for $A_p \neq 0$ and $p \neq 2$.

It is now sufficient to prove that any ring A with condition (I) and with an additive elementary 2-group is a restricted Boolean ring. In fact, condition (I) implies $a^2 = a$ for any $a \in A$ and

$$ab = ba \in Ia + Ib$$

for any a and b of A.

Equality $ab = a + b$ cannot occur for $a \neq 0$. Indeed, assuming $ab = a + b$, the equations

$$ab = a(ab) = a(a + b) = a + ab = a + a + b = b$$

would yield the contradiction $a = 0$ with the assumption $a \neq 0$. But $a^2 = a$ and $ab = ba \neq a + b$ for any $a, b \in A$ mean that $A = A_2$ is a restricted Boolean ring.

Hence the implication (I) $\Rightarrow$ (II) holds.

Conversely, assume that A is a ring with condition (II). Let S be an arbitrary additive subgroup of A. According to condition (II), S has an additive direct decomposition $S = S_2 + \sum_p S_p$, where p is an odd prime number. Consequently, $S_p S_q = S_q S_p = 0$ for $p \neq q$. Since, for any non-zero p-component S_p, $S_p \cong I/(p)$, there must be $S_p^2 = S_p$, whence also

$$\Big(\sum_p S_p\Big)^2 = \sum S_p.$$

Since $a^2 = a$ for any $a \in A_2$, we infer, by the definition of the product BC of subsets B and C of A, that the 2-component S_2 of S satisfies $S_2^2 \supseteq S_2$. On the other hand, also $S_2^2 \subseteq S_2$ holds, because the 2-component A_2 of A is a restricted Boolean ring. Therefore $S_2^2 = S_2$, whence $S^2 = S$.

Consequently, we have also the implication (II) $\Rightarrow$ (I), which completes the proof.

Examples 5. (1) Let A be the algebra over the field of two elements, generated by the elements a, b and c with the table of multiplication

	a	b	c
a	a	c	c
b	c	b	c
c	c	c	c

Then A is a Boolean ring having eight elements such that the subgroup $S = Ia + Ic$ is an idempotent subring, but the subgroup $T = Ia + Jb$, satisfying $T^2 = A \neq T$, is not a subring and is not idempotent. Therefore A is a Boolean (but not restricted Boolean) ring without condition (I).

(2) Let A be the complete direct sum of the fields $K_{2,n}$ of two elements, $n = 1, 2, 3, \ldots$ Furthermore, let a_n be the infinite vector, treated as an element in A, which has 0 in the first n components and 1 elsewhere. Let b_n denote the product $a_1 a_2, \ldots, a_n$ of A. Then A is a (restricted) Boolean ring, which is also strongly regular, but the infinite proper descending chain of principal ideals

$$(b_1) \supset (b_2) \supset (b_3) \supset \ldots$$

shows that A is not an MPR-ring. Obviously, b_n is the unity element of the ideal (b_n). Let C_n be an ideal of A such that the direct decomposition

$$(b_{n-1}) = (b_n) \oplus C_n$$

holds for any $n \geqslant 2$. Construct the direct sum $D = \sum_{n \geqslant 2} \oplus C_n$. The ideal D of A lies in the principal ideal (b_1) of the (commutative) ring A of cardinality continuum, and the ring D does not contain unity element (cf. Satz 2.5 of part II of [15]).

(3) Let A be the direct sum of two fields of two elements, that is $A = Ia + Ib$ with

$$2A = ab = ba = a^2 - a = b^2 - b = 0.$$

Then A satisfies condition (I). Consequently, A is a restricted Boolean ring. The subgroup $K = I(a+b)$ is a subring, but K is neither an ideal, nor a (ring theoretical) direct summand of A.

(4) Let A be the direct sum of a field $B = Ib$ of order two and of a ring $C = Ic$ of order two with $c^2 = 0$. Then A does not satisfy condition (I), any subring is a (ring-theoretical) direct summand of A, but the subgroup $I(b+c)$ is not a two-sided ideal of A.

(5) Let A be the ring Ia with $a^2 = 0$. Then A is an infinite cyclic ring in which any additive subgroup is a twosided ideal with trivial multiplication, A does not satisfy condition (I), and $2A$ is not a direct summand of A.

Remarks 6. (1) Let C_1 denote the class of all rings with condition (I). In the author's paper [14] there is determined the class C_2 of all rings such that any subring is a (ringtheoretical) direct summand. Furthermore, Rédei [13] has determined the class C_3 of all rings such that any additive subgroup is a two-sided ideal. These latter rings are called *full ideal rings*. Now, example (3) shows that $C_1 \nsubseteq C_2$ and $C_1 \nsubseteq C_3$. Furthermore, example (4) yields $C_2 \nsubseteq C_1$ and $C_2 \nsubseteq C_3$. Finally, by example (5), we have also $C_3 \nsubseteq C_1$ and $C_3 \nsubseteq C_2$. Consequently, C_1, C_2 and C_3 are different classes of rings.

(2) In the proof of the implication (I) $\Rightarrow$ (II), the theorem of Forsythe and McCoy [4], according to which any regular ring without non-zero nilpotent elements is a subdirect sum of division rings, was not used.

(3) We have seen that $a = ma^2$ for any $a \in A$, where m is an integer and A is a ring with condition (I). This means that $a \in Ia^2$ for any $a \in A$. Let $C(a)$ denote the subgroup Ia^2. Condition (I) implies the C-regularity $a \in C(a)$ for any $a \in A$, which satisfies $C(a\varphi) = (C(a))\varphi$ for any (ring-theoretical) homomorphism φ of A. The axiom P_1 of Brown and McCoy [1], p. 302, holds, but the axiom P_2 generally fails to be satisfied for $C(a)$. On the other hand, this $C(a)$ is a modified form of $F(a)$ of example 4 of [1]; p. 308, for which axioms P_1 and P_2 are already satisfied in any ring A, treated as a (F, Ω)-group.

(4) The upper radical R (cf. Divinsky [3]), defined by the class C_1 of all rings with condition (I) has the property that any homomorphic image of any R-semisimple ring is again R-semisimple.

(5) It would be interesting to investigate the question, whether any finitely generated additive subgroup of any ring with condition (I) is, or is not, a direct sum of finite prime fields. **(P 782)**

REFERENCES

[1] B. Brown and N. H. McCoy, *Some theorems on groups with application to ring theory*, Transactions of the American Mathematical Society 69 (1950), p. 302-311.

[2] A. H. Clifford, *Bands of semigroups*, Proceedings of the American Mathematical Society 5 (1954), p. 499-504.

[3] N. Divinsky, *Rings and radicals*, London 1965.

[4] A. Forsythe and N. H. McCoy, *On the commutativity of certain rings*, Bulletin of the American Mathematical Society 52 (1946), p. 523-526.

[5] А. И. Герчиков, *Кольца, разложимые в прямую сумму тел*, Математический Сборник 7(49) (1940), p. 591-597.

[6] N. Jacobson, *Structure of rings*, Providence 1964.

[7] T. Kandô, *Strong regularity in arbitrary rings*, Nagoya Mathematical Journal 4 (1952), p. 51-53.

[8] I. Kaplansky, *Infinite abelian groups*, Ann Arbor 1954.

[9] L. G. Kovács, *A note on regular rings*, Publicationes Mathematicae Debrecen 4 (1956), p. 465-468.

[10] S. Lajos, *A characterization of semigroups, which are semilattices of groups* (to appear).

[11] — and F. Szász, *Characterizations of strongly regular rings, I*, Proceedings of Japan Academy 46 (1970), p. 38-40; *II*, ibidem 46 (1970), p. 287-289.

[12] N. H. McCoy, *The theory of rings*, New York 1964.

[13] L. Rédei, *Die Vollidealringe*, Monatshefte für Mathematik 56 (1952), p. 89-96.

[14] Ф. Сас, *О кольцах, каждое подкольцо которых является прямым слагаемым кольца*, Математический Сборник 40 (82) (1956), p. 269-272.

[15] F. Szász, *Über Ringe mit Minimalbedingung für Hauptrechtsideale, I*, Publicationes Mathematicae Debrecen 7 (1960), p. 54-64; *II*, Acta Mathematica Academiae Scientiarum Hungaricae 12 (1961), p. 417-439; *III*, ibidem 14 (1963), p. 447-461.

[16] — *Generalized biideals of rings, I*, Mathematische Nachrichten 47 (1970), p. 355-360; *II*, ibidem 47 (1970), p. 361-364.

[17] — *A class of regular rings*, Monatshefte für Mathematik 75 (1971), p.168-172.

Reçu par la Rédaction le 29. 6. 1970