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SOME MEASURES DETERMINED BY MAPPINGS OF THE CANTOR SET

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0. In this note we present a function-space technique for constructing measures on the real line. These measures are just the distribution functions of functions on the Cantor set, with respect to the conventional product measure. We obtain, first of all, measures λ all of whose powers are mutually singular; naturally we insist that $\lambda(E) \equiv \lambda(-E)$. Then we derive some of the results of Hewitt and Kakutani [2]. In several places we have proved that a certain phenomenon occurs on a dense G_δ-set, rather than proving only the existence.

1. Let X denote the Cantor set, and its elements be denoted by $x = (x_1, x_2, x_3, \ldots)$, $x_i = 0, 1$ $(1 \leqslant i \leqslant \infty)$. With component-wise addition modulo 2, X is a compact topological group, whose Haar measure is denoted by μ. For a sequence $b_1, b_2, b_3, \ldots$ of positive numbers with finite sum there is defined the continuous function

$$\varphi(x) = \sum_1^\infty b_i x_i, \qquad x \epsilon X.$$

Now write $\lambda(B) = \mu\big(\varphi^{-1}(B)\big)$ for every Borel subset of the real numbers; sometimes we write λ_φ for λ.

THEOREM 1. *Suppose that r and s are integers and $s > r \geqslant 1$. If* $\limsup b_{i+1}/b_i < T$, *where*

$$\sum_1^\infty T^i = (r+s)^{-1},$$

then the r-fold convolution, λ^r, of λ_φ, is singular to every translate of λ^s.

Proof. If the first m of the numbers $b_1, b_2, \ldots$ are replaced by 0, the measure λ is replaced by λ', say, but it is clear that $\lambda = \lambda' * \sigma$ for some measure σ with finite support. Thus we can suppose at the start that $b_{i+1}/b_i < T$ $(1 \leqslant i < \infty)$. The last inequality is intended to justify some elementary arguments from probability theory, and we write so as to exploit this. λ^s is supported in the set of numbers of the form $\sum_1^\infty b_i X_i$, where $0 \leqslant X_i \leqslant s$ $(1 \leqslant i < \infty)$. Inasmuch as

$$b_j > s \sum_{j+1}^{\infty} b_i \quad \text{for each} \quad j \geqslant 1,$$

the multipliers $X_1, X_2, X_3, \ldots$ are well-defined continuous functions of the sum. λ^s is determined by the condition that the X's are jointly independent, and assume each integral value k in $[0, s]$ with probability $2^{-s}\binom{s}{k}$. The measure λ^r is supported in the set of numbers of the form $\sum_1^\infty b_i Y_i$, $0 \leqslant Y_i \leqslant r$ $(1 \leqslant i < \infty)$. It remains to prove that any translate of this latter set has λ^s measure 0; let it be translated by a real number c.

For each integer $N \geqslant 1$, consider those N-tuples $(X_1, \ldots, X_N)$ which can be associated to an element in the given translate, as well as in the support of λ^s; we claim that at most r^N vectors can be so realized. Suppose, indeed, that

$$\sum b_i X_i = c + \sum b_i Y_i \quad \text{and} \quad \sum b_i X_i' = c + \sum b_i Y_i',$$

where $(X_1, \ldots, X_N) \neq (X_1', \ldots, X_N')$. To establish our claim it is enough to deduce that then $(Y_1, \ldots, Y_N) \neq (Y_1', \ldots, Y_N')$. In the opposite case let M be the first index at which $X_i \neq X_i'$, so we have

$$b_M \leqslant b_M |X_M - X_M'| \leqslant r \sum_{N+1}^{\infty} b_i + s \sum_{M+1}^{\infty} b_i < b_M (r+s) \sum_{1}^{\infty} T^i \leqslant b_M,$$

a contradiction.

Now there are $(s-1)^N \geqslant r^N$ N-tuples $(X_1, \ldots, X_N)$ in which no coordinate is 0, and these $(s-1)^N$ choices have the maximum total probability for any collection of $(s-1)^N$ N-tuples; this probability is $(1-2^{-s})^N$. This proves Theorem 1.

Remark. Let $a = \sum_1^\infty b_i$, so that for each Borel set $B, \varphi^{-1}(-B) = -\varphi^{-1}(a+B)$, whence $\lambda(-B) = \lambda(a+B)$. Writing $\lambda^\sim(B) \equiv \lambda(-B)$ we see that Theorem 1 holds equally for $\lambda + \lambda^\sim$.

COROLLARY. *If* $\lim b_{i+1}/b_i = 0$, *distinct convolution powers of* $\lambda + \lambda^\sim$ *are singular.*

This is not surprising in view of [1], 3.8, since the support of the measure λ is close to being independent. It is possible nevertheless to replace the precipitously decreasing sequence $b_1, b_2, b_3, \ldots$ by one which decreases irregularly but not too quickly. This requires some technical remarks.

LEMMA 2. *Let $P(R)$ be the space of Borel probability measures on the real numbers R, in its w^* (dual-space) topology. Then the subset $S = \{(\varrho, \sigma) \in P \times P : \varrho \perp \sigma\}$ has type G_δ in $P \times P$.*

Proof. For every $t \in [0, 2)$, $\|\varrho - \sigma\| > t$ if and only if there is a continuous function f, having compact support in R, such that $|\langle \varrho - \sigma, f \rangle| > t\|f\|$. The union for all f is an open subset of $P \times P$; S is the intersection of these sets, for t rational in $[0, 2)$.

We introduce now a Cantor set C whose elements are written $\xi = (\xi_1, \xi_2, \xi_3, \ldots)$, $\xi_i = 0, 1$ $(1 \leqslant i \leqslant \infty)$. Making use of a fixed sequence $b_1, b_2, b_3, \ldots$ of positive numbers with a finite sum, each ξ defines a continuous function on the space X according to the formula

$$\xi(x) = \sum_1^\infty \xi_i x_i b_i .$$

THEOREM 3. *For each ξ in C let $\lambda_\xi = \mu \circ \xi^{-1}$. Then the subset of C $\{\xi \in C : (\lambda_\xi + \lambda_\xi^\sim)^s \perp (\lambda_\xi + \lambda_\xi^\sim)^r$ for $s > r \geqslant 1\}$ is a dense G_δ in C.*

Proof. It is clear that the mappings $\xi \to 2^{-s}(\lambda_\xi + \lambda_\xi^\sim)^s$, $s \geqslant 1$, are continuous on C into $P(R)$, so that the subset in question has type G_δ in C. It contains a dense subset, for example all ξ for which $\xi_i = 0$ outside a finite set, except for a remainder which is sufficiently lacunary. Theorem 3 is thus proved, in view of the corollary.

An interesting special case is based upon the fact that the sequences in C which contain infinitely many pairs of neighboring 1's form a dense G_δ, hence intersect the set constructed in Theorem 3. Let now $\sum_1^\infty b_i < \infty$ but $b_{i+1}/b_i \to 1$. For a sequence ξ in C which has infinitely many pairs of neighboring 1's, the non-zero members of the sequence $\xi_1 b_1, \xi_2 b_2, \ldots$, say b'_j, have the property that

$$\limsup b'_{j+1}/b'_j = 1 .$$

It is natural to ask whether the dense subset of C constructed in Theorem 3 can contain elements ξ for which

$$\lim_{N \to \infty} \frac{1}{N} \sum_1^N \xi_i = 1 .$$

In the following paragraphs we show that this is indeed possible (and, in a sense, typical) for a rather special sequence $b_1, b_2, b_3, \ldots$

THEOREM 1*. *Let $b_1, b_2, b_3, \ldots$ be non-increasing powers of $\frac{1}{2}$. Suppose there is a sequence of integers*

$$0 = k(0) < k(1) < \ldots < k(n) < k(n+1)$$

such that

$$\sum_{n=1}^{\infty} 2^{sk(n)-sk(n+1)} = \infty, \qquad (r+s)\sum_{k(n)+1}^{\infty} b_i < b_{k(n)}, \qquad n = 1, 2, 3, \ldots$$

Then λ_φ^s is singular to every translate of λ_φ^r. (Here s, r, and λ_φ are as in Theorem 1.)

Proof. λ^s is supported by numbers of the form

$$\sum_1^{\infty} b_i X_i, \qquad 0 \leqslant X_i \leqslant s \qquad (1 \leqslant i < \infty).$$

Writing

$$Y_N = \sum_1^{k(N)} b_i X_i,$$

we find that two distinct values in the range of Y_N differ by $>(r+s)\sum_{k(N)+1}^{\infty} b_i X_i$, so that the functions Y_N are uniquely determined by the sum $\sum_1^{\infty} b_i X_i$. The same is then true for the sums $Z_1 = Y_1$, $Z_2 = Y_2 - Y_1, \ldots, Z_N = Y_N - Y_{N-1}$. For each $N > 1$ there are $(1+s)^{k(N)}$ possible values for the n-tuple $(Z_1, \ldots, Z_N)$, each occurring with λ^s-probability at least $2^{-sk(N)}$. As in Theorem 1, we find, from this, that the λ^s probability of any set

$$\left\{c + \sum_1^{\infty} b_i X_i,\ 0 \leqslant X_i \leqslant r\right\}$$

is at most

$$\prod_1^{\infty} \{1 - 2^{s[k(N)-k(N+1)]}\}.$$

Since this diverges to 0, the proof is achieved.

Let C_1 be the subset of C of sequences ξ such that

$$\lim_{N\to\infty} \frac{1}{N}\sum_1^N \xi_i = 1.$$

C_1 is a set of the first category in itself, so we introduce a stronger topology *via* the metric

$$d(\xi, \xi') \equiv \sup_{1\leqslant N<\infty} \left| \frac{1}{N}\sum_1^N \xi_i - \xi_1' \right|;$$

C_1 is complete in the topology of the metric d.

THEOREM 3*. *If* $b_1, b_2, b_3, \ldots$ *is a sequence of strictly decreasing powers of* $\frac{1}{2}$, *then the set*

$$\{\xi \in C_1 : (\lambda_\xi + \lambda_{\tilde{\xi}})^2 \perp (\lambda_\xi + \lambda_{\tilde{\xi}})^r \text{ for } s > r \geqslant 1\}$$

is a dense G_δ *in* C_1.

Proof. By Theorem 3*, $(\lambda_\xi + \lambda_{\tilde{\xi}})^s \perp (\lambda_\xi + \lambda_\xi)^r$ if it contains blocks of the form $A_1 B_n A_2$, where A_1 and A_2 are strings of at least $1 + \log 2s$ 0's, B_n is a block of length $\leqslant \log n / s \log 2$ which contains at least a single digit 1. In fact, if we suppress the 0's from the sequence $\xi_1 b_1, \xi_2 b_2, \ldots$, we obtain a sequence of decreasing powers of $\frac{1}{2}$ that is allowed by Theorem 1*. Any sequence in C_1 can be transformed into a sequence of the foregoing sort by introducing (when necessary) the blocks of 0's. It is plain that this can be done so that the transformed sequence has arbitrarily small d-distance from the given one. The theorem is thereby proved.

2. The functional notation

$$\xi(x) = \sum_1^\infty \xi_i b_i x_i$$

is retained, and the numbers $b_1, b_2, b_3, \ldots$ are required to be rational. This condition and the construction below should be compared with [2], (4.3) and (4.4). We denote by A the set of sequences ξ in C with the property that for each complex number z of modulus $\leqslant 1$ there is a sequence $m_1, m_2, m_3, \ldots$ of integers for which

$$\lim_{k \to \infty} \int_X \exp 2\pi i m_k \xi(x) \mu(dx) = z \int \chi(x) \mu(dx)$$

for each continuous character χ of X.

The application of this, set forth in [2], is as follows. For each complex number z of modulus $\leqslant 1$, there is a sequence of continuous characters σ_n such that

$$\lim_{n \to \infty} \int \sigma_n(t) f(t) \lambda_\xi(dt) = z \int f(t) \lambda_\xi(dt)$$

for each bounded Borel function f on $(-\infty, \infty)$.

Remark. We can describe a sequence $O(k, l)$ of open subsets of C whose intersection is exactly A. Let $\chi_1, \chi_2, \chi_3, \ldots$ be the non-trivial continuous characters of X, $V_1, V_2, V_3, \ldots$ a sequence of open subsets of the complex unit disk which forms a basis for its topology.

Say that $\xi \in O(k, l)$ if there is some integer m such that

$$\int \exp 2\pi i m \xi(x) \mu(dx) \in V_k$$

and

$$\left|\int \exp 2\pi im\,\xi(x)\chi_j(x)\mu(dx)\right| < (1+l)^{-1}$$

for $1 \leqslant j \leqslant l$ $(1 \leqslant k,\ l < \infty)$. It is evident that $A \supseteq \bigcap O(k, l)$, and the converse is proved by Cantor's diagonal process.

TECHNICAL LEMMA. *Let $c_1, \ldots, c_r$ be rational numbers, and $d_1, d_2, d_3, \ldots$ a sequence of positive rational numbers convergent to* 0. *For each $\varepsilon > 0$, and each integer $s > 1$, there is a subset of distinct numbers $d_1^*, \ldots, d_s^*$ among the d's with the following property.*

For any real numbers $a_1, \ldots, a_s$, there is an integer m such that

$$mc_i \equiv 1 (\text{modulo} 1), \quad 1 \leqslant i \leqslant r,$$

$$|md_i^* - a_i| < \varepsilon (\text{modulo} 1), \quad 1 \leqslant i \leqslant r.$$

Proof. If the numbers d_i are expressed in lowest terms, $d_i = p_i/q_i$ with p_i and q_i relatively prime, then $q_i \to \infty$ because $d_i \to 0$. Thus we can choose d_1' so that its denominator is $> \varepsilon^{-1} \cdot$ {the least common multiple of the denominators of the c's} and each d_i^* to have denominator $> \varepsilon^{-1} \cdot$ {the least common multiple of all previous denominators}. Then there exist integers $m_1, \ldots, m_s$ such that

$$m_i c_i \equiv 0 (\text{modulo} 1), \quad 1 \leqslant i \leqslant r,$$

$$m_j d_j^* \equiv 0 (\text{modulo} 1), \quad j < i,$$

and the denominator, in lowest terms, of $m_i d_i^*$ is $> \varepsilon^{-1}$. We can choose inductively integers $n_1, \ldots, n_s$ such that

$$|(n_1 m_1 + \ldots + n_i m_i) d_i^* - a_i| < \varepsilon.$$

Then

$$|(n_1 m_1 + \ldots + n_s m_s) d_i^* - a_i| < \varepsilon (\text{modulo} 1).$$

THEOREM 4. *A is dense in C.*

Proof. We must prove that each of the open sets described in the previous remark is dense. Let $\xi_1, \ldots, \xi_h$ be arbitrarily assigned elements of $\{0, 1\}$ and let us suppose for simplicity that all the characters in question have the form

$$x \to \exp \pi i (\varepsilon_1 x_1 + \ldots + \varepsilon_h x_h), \quad \varepsilon_j \in \{0, 1\},\ 1 \leqslant j \leqslant h.$$

Then

$$\int\limits_X \chi(x) \exp 2\pi im\xi(x)\mu(dx)$$

$$= \prod_1^h \left\{\frac{1}{2} + \frac{1}{2}\exp \pi i [\varepsilon_k + 2mb_k \xi_k]\right\} \cdot \prod_{h+1}^{\infty} \left\{\frac{1}{2} + \frac{1}{2}\exp 2\pi imb_k \xi_k\right\}.$$

Let $a_1, \ldots, a_s$ be arbitrary real numbers, with s also arbitrary, and in the technical lemma, take $b_1, \ldots, b_h$ for the c's and $b_{h+1}, b_{h+2}, \ldots$ for the d's. When $d_1^*, \ldots, d_s^*$ have been chosen, dependent only upon s and ε, we choose m as specified in the lemma. Then set $\xi_k = 1$ for $k > h$ exactly when b_k is one of the d^* selected. The first $\prod$ vanishes unless $\varepsilon_1 = \ldots = \varepsilon_h = 0$, and in that case the integral is

$$\prod^{*}\left\{\frac{1}{2}+\frac{1}{2}\exp 2\pi i m b_k\right\},$$

the product $\overset{*}{\prod}$ being taken for the s distinct k's for which b_k is chosen as a d^*. This product can be made arbitrarily close to any product

$$\prod_{j=1}^{s}\left\{\frac{1}{2}+\frac{1}{2}\exp 2\pi i a_j\right\},$$

and hence ([2], (4.1)), by increasing s, arbitrarily close to any number in the unit disk. The proof is complete.

Remark. Kronecker's Theorem on simultaneous approximation (modulo 1) can be used to obtain a variant of Theorem 4 in the case that the numbers $b_1, b_2, b_3, \ldots$ are irrational and linearly independent modulo the integers. Further variants can be obtained by adapting the technical lemma to sequences $b_1, b_2, b_3, \ldots$ of mixed types.

REFERENCES

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