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MAPPINGS ON COMPACT METRIC SPACES

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1. Introduction. In this paper, we consider only those spaces which are separable and metrizable.

Let M be a compact space and f a continuous mapping of M. The classical Hurewicz theorem on dimension lowering mappings (see [4], p. 67) then concludes that

(i) $$\dim M \leqslant \dim f(M) + \dim f,$$

where

$$\dim f = \sup\{\dim f^{-1}(y) \mid y \in f(M)\}.$$

Let

$$A = \{y \in f(M) \mid \dim f^{-1}(y) \geqslant \dim M - \dim f(M)\}.$$

An equivalent formulation for the conclusion (i) is that

(ii) $$A \neq \emptyset.$$

Moreover, in case f is non-constant and M is a Cantor-manifold, we have shown that

(iii) $$\dim A > 0$$

(see [2] or [3]). Thus a natural question arises as to how large is the dimension of A. Here we will give a lower bound for the dimension of A. Moreover, this will come out as a direct consequence of a result which gives a sharper estimate for the dimension of M than that given in inequality (i).

2. Main results. We shall begin by stating two lemmas. The first can be found in either [2] or [3] and the second is a trivial consequence of a theorem of Nishiura [5].

2.1. LEMMA. *Let f be a continuous mapping of a compact space M. For any integer $m \geqslant -1$, the set*

$$A_m = \{y \in f(M) \mid m \leqslant \dim f^{-1}(y)\}$$

is F_σ in $f(M)$.

2.2. LEMMA. *Let N be a space of finite dimension and X an F_σ subset of N. Then for any two disjoint closed subsets B_1 and B_2 of N, there is an open neighborhood W of B_1 in N such that*

$$W \cap B_2 = \emptyset, \quad \dim \mathrm{Fr}(W) \leqslant \dim N - 1,$$

and

$$\dim[X \cap \mathrm{Fr}(W)] \leqslant \dim X - 1,$$

where $\mathrm{Fr}(W)$ *denotes the frontier of* W.

2.3. THEOREM. *Let M be a compact space, f a continuous mapping of M and let*

$$A = \{y \in f(M) \,|\, \dim f^{-1}(y) \geqslant \dim M - \dim f(M)\}.$$

Then

$$\dim M \leqslant \dim A + \dim f.$$

Proof. If $\dim f(M) \geqslant \dim M$, then $A = f(M)$ and the assertion follows. Thus we may suppose that $\dim M > \dim f(M)$. In this case, $\dim A \geqslant 0$ which follows from the Hurewicz theorem. The further proof is by induction on $\dim f(M)$, holding $\dim f$ fixed.

If $\dim f(M) = 0$, then the Hurewicz theorem implies that

$$\dim M \leqslant 0 + \dim f \leqslant \dim A + \dim f.$$

Now assuming the assertion for $\dim f(M) \leqslant m$, we shall show it for $\dim f(M) = m+1 < n$, where $n = \dim M$.

Since every compact space contains a Cantor-manifold of the same dimension (see [1], Th. VI, 8, p. 94), we may assume, without loss of generality, that M is an n-dimensional Cantor-manifold. Since $\dim f^{-1}(y) \leqslant \dim f \leqslant \dim f + \dim A$ for any $y \in f(M)$ and since $M = \bigcup \{f^{-1}(y) \,|\, y \in f(M)\}$, the assertion will follow from ([1], VI, 4. G, p. 90) if we can verify the remaining condition of that proposition. Suppose U is an open set in M containing $f^{-1}(y)$. (Clearly, we may assume $\overline{U} \neq M$). Then $f(M-U)$ is a closed subset of $f(M)$. Since the set A is F_σ in $f(M)$ by Lemma 2.1, it follows from Lemma 2.2 that there is an open set W in $f(M)$ such that

1° $y \in W$, $W \cap f(M-U) = \emptyset$,

2° $\dim \mathrm{Fr}(W) \leqslant \dim f(M) - 1 = m$

and

3° $\dim[A \cap \mathrm{Fr}(W)] \leqslant \dim A - 1$.

Condition 1° implies that

$$f^{-1}[W] \cap (M-U) = \emptyset,$$

i.e.,

$$f^{-1}[W] \subset U.$$

Moreover, $f^{-1}[W]$ is open in M, as an inverse image under a continuous map of an open set, and contains $f^{-1}(y)$. Let

$$V = f^{-1}[W] \quad \text{and} \quad B = f^{-1}[\mathrm{Fr}(W)].$$

Clearly $\overline{V} = \overline{f^{-1}[W]} \subset f^{-1}[\overline{W}]$; so that

4° $\mathrm{Fr}(V) \subset B$.

Let g be f restricted to B and let

$$A^* = \{z \in g(B) \mid \dim g^{-1}(z) \geqslant \dim B - \dim g(B)\}.$$

Since we assumed $\overline{U} \neq M$, B clearly separates the n-dimensional Cantor-manifold and

5° $\dim B \geqslant \dim M - 1 = n-1$.

Moreover, by 2°,

6° $\dim g(B) = \dim \mathrm{Fr}(W) \leqslant \dim f(M) - 1 = m < n-1$.

Hence g is a dimension lowering map of B onto $g(B) = \mathrm{Fr}(W)$. Applying the induction hypothesis, we obtain

7° $\dim B \leqslant \dim A^* + \dim g$.

Obviously, we have $\dim g \leqslant \dim f$; and since

$$\dim B - \dim g(B) \geqslant (\dim M - 1) - \big(\dim f(M) - 1\big) = \dim M - \dim f(M)$$

by inequalities 5° and 6°, we also have

8° $A^* \subset A \cap \mathrm{Fr}(W)$.

Hence, it follows from the inequalities 3°, 4°, 7°, and 8° that

$$\dim \mathrm{Fr}(V) \leqslant \dim B \leqslant \dim A^* + \dim g \leqslant \dim A - 1 + \dim f.$$

Thus the remaining condition of ([1], VI, 4. G, p. 90) is satisfied and the assertion follows.

2.4. Remark. Examples can easily be constructed showing that the estimate for the dimension of M given by Theorem 2.3 is better than that given by the classical Hurewicz theorem.

Also, we note the condition that M be compact in Theorem 2.3 can be replaced by the weaker requirement that M contains a compact subset C with $\dim C = \dim M$. Consequently, Theorem 2.3 is applicable to continuous mappings on manifolds. In fact, a slightly more general form of Theorem 2.3 holds.

2.5. Theorem. *Let M be a locally compact space, f a perfect mapping (= continuous closed mapping such that $f^{-1}f(x)$ is compact for every $x \in M$) of M and let*

$$A = \{y \in f(M) \mid \dim f^{-1}(y) \geqslant \dim M - \dim f(M)\}.$$

Then

$$\dim M \leqslant \dim A + \dim f.$$

Proof. Since f is perfect and M is locally compact, $f(M)$ is locally compact. Hence both M and $f(M)$ have metrizable one-point compactifications $\tilde{M}$ and $\big(f(M)\big)^{\sim}$, respectively. Applying the perfectness of f again, we obtain a continuous mapping g of $\tilde{M}$ onto $\big(f(M)\big)^{\sim}$ which is an extension of f. Let

$$A^* = \{y \in \big(f(M)\big)^{\sim} \mid \dim g^{-1}(y) \geqslant \dim \tilde{M} - \dim \big(f(M)\big)^{\sim}\}.$$

It follows from Theorem 2.3 that

$$\dim \tilde{M} \leqslant \dim A^* + \dim g.$$

Since clearly $\dim g = \dim f$ and since the adjunction of a single point to a set does not raise the dimension of the given set, we have

$$\dim M \leqslant \dim A + \dim f.$$

The theorem is now proved.

The following corollary is an easy consequence of Theorem 2.5:

2.6. COROLLARY. *Let f be a perfect mapping of a locally compact space M and let*

$$A = \{y \in f(M) \mid \dim f^{-1}(y) \geqslant \dim M - \dim f(M)\}.$$

Then

(i) *if* $0 \leqslant \dim f \leqslant \dim M - k$, *then* $\dim A \geqslant k$,

and

(ii) *if* $0 \leqslant \dim A \leqslant \dim M - k$, *then* $\dim f \geqslant k$.

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