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ON EIGENVECTORS OF COMPACT CONTRACTIONS ACTING IN LINEAR METRIC SPACES

BY

J. HALPERIN, Jr. (TORONTO), S. ROLEWICZ (WARSZAWA) AND A. SHIELDS (ANN ARBOR, MICH.)

Let X be a complete linear metric space over the complex numbers. We say that a linear operator T is *compact* if there is a neighborhood U of the origin such that the set $\overline{TU}$ is compact (see [6] and [11]).

We say that an element $x \in X$ is an *eigenvector* if there is a number $\lambda \neq 0$, called an *eigenvalue*, such that $Tx = \lambda x$.

Let X be a Banach space. We can introduce a norm for the operator T in the usual manner. It follows from the general theory of Banach algebras (see [4], for example) that the limit $\lim_n (\|T^n\|)^{1/n}$ always exists. Obviously if the operator T has an eigenvector, then this limit is different from zero. On the other hand, if this limit is different from zero, then the general theory of Banach algebras implies that there is a number $\lambda \neq 0$ such that the operator $T - \lambda I$ is not invertible. Hence from Riesz's theory of compact operators (see [7] and [8]) it follows that λ is an eigenvalue and there is an eigenvector x.

Now let X be a locally bounded space. Using the results of papers [1] and [9] we can introduce a p-homogeneous (1) norm in the space X in such a way that the norm determines a topology of X. The norm of the operator T can be defined in the same way as in the case of Banach spaces: $\|T\| = \sup \|Tx\|$ $(\|x\| \leqslant 1)$.

Using the theory of locally bounded algebras founded and developed by W. Żelazko (see [12]), and the general Riesz theory (see [11]), we are able to carry over the argument from the Banach case to the locally bounded case and so we obtain the result that an eigenvector exists if and only if $\lim_n (\|T^n\|)^{1/n} \neq 0$.

In the general case the theory of metric algebras cannot be used. Here we formulate a sufficient condition for the existence of eigenvectors.

(1) The norm is called *p-homogeneous* $(0 < p \leqslant 1)$ if $\|tx\| = |t|^p \|x\|$.

THEOREM. *Let X be a complete linear metric space with metric $\varrho(x, y)$ (not necessarily translation invariant). Let T be a compact linear operator acting in X. Assume moreover that T is a contraction, that is, $\varrho(Tx, Ty) \leqslant \varrho(x, y)$. If there is a compact set E that is invariant under T (that is, $TE \subset E$) and does not contain 0, then the operator T has an eigenvector.*

The proof of this theorem is based on the following proposition proved by Freudenthal and Hurewicz [3]:

PROPOSITION. *Let K be a compact metric space. Let T be a continuous function mapping K onto itself. If T is a contraction, then T is an isometry, that is, $\varrho(Tx, Ty) = \varrho(x, y)$.*

Proof of the theorem. Let E be an invariant compact set that does not contain zero. Let $E_1 = \bigcap T^n E$. The set E_1 is a non-empty compact set and, moreover, T transforms E_1 onto itself. Let

$$K = \bigcup_{|\lambda| \leqslant 1} \lambda E_1.$$

The set K is also compact and T transforms K onto itself. Now let $K^n = K + K + \ldots + K$ (n-fold vector sum). Each of the sets K^n is compact and for each $n = 1, 2, \ldots$ the operator T transforms K^n onto itself. Hence, by the above proposition, T is an isometry on each of the sets K^n. Therefore it is also an isometry on the subspace

$$X_0 = \overline{\bigcup_n K^n}$$

(the closed subspace spanned by the set E_1).

But T is a compact operator and hence (see [2]) X_0 is finite dimensional. Since T transforms X_0 onto itself, T must have an eigenvector, q. e. d.

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UNIVERSITY OF TORONTO
INSTITUTE OF MATHEMATICS OF THE POLISH ACADEMY OF SCIENCES
UNIVERSITY OF MICHIGAN

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