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REMARKS ON EIGENVALUES OF INFINITE MATRICES

BY

WOJBOR A. WOYCZYŃSKI (WROCŁAW)

1. Let X be a *complex linear sequence space*, i.e. the linear space having sequences $\boldsymbol{x} = \{x_i\}_{i=1,2,\ldots}$ of complex numbers ($x_i \in C$) as its elements. By $\mathbf{1}_i$ we denote the sequence having 1 on the i-th place, and zeros besides.

Let now A be a linear operator acting from X into itself represented by an infinite complex matrix $(a_{ij})_{i,j=1,2,\ldots}$. Thus the acting $A\boldsymbol{x} = \boldsymbol{y}$ ($\boldsymbol{x} = \{x_i\}$, $\boldsymbol{y} = \{y_i\}$) of the operator A may be described by means of the formula

$$y_i = \sum_{j=1}^{\infty} a_{ij}\, x_j, \qquad i = 1, 2, \ldots$$

A complex number λ is said to be an *eigenvalue of A relative to Y* (suppose Y to be a subset of X) if and only if there is $y \in Y$, $y \neq \mathbf{0}$, such that the equation $A\boldsymbol{y} = \lambda \boldsymbol{y}$ holds.

The purpose of this note is to prove the following

THEOREM. *Let $A = (a_{ij})_{i,j=1,2,\ldots}$ be an infinite matrix with $a_{ii} \neq 0$, and suppose that there exists an Orlicz function Φ such that*

$$\sum_{i=1}^{\infty} \Psi\left(\left\| \left\{\frac{a_{ij}}{a_{ii}}\right\}_{j=1,2,\ldots} - \mathbf{1}_i \right\|_{\Phi}\right) \leqslant 1.$$

Then $\lambda = 0$ is not an eigenvalue of A relative to the sequence Orlicz space l_Ψ, where the Orlicz function Ψ is complementary to Φ in the sense of Young.

Terms used above will be explained later on.

From this theorem we immediately get the evaluation

$$\sum_{i=1}^{\infty} \Psi\left(\left\| \left\{\frac{a_{ij}}{a_{ii} - \lambda}\right\}_{j=1,2,\ldots} - \mathbf{1}_i \right\|_{\Phi}\right) > 1$$

for any eigenvalue λ of A relative to the Orlicz space l_Ψ. Special cases of such an estimation can find an application in approximative determination of eigenvalues of certain operators occurred in quantum mechanics and other branches of physics. Here we refer to [5], where a number of interesting examples have been considered. Our theorem gives also the simultaneous generalization of some theorems of papers [2]-[5], [7] and [8]. Papers [2]-[4], [7] and [8] were concerning with finite square matrices and they contain essentially a multiple rediscovery of XIX century L. Lévy's theorem. Their results can be shortly formulated in the following way: if $A = (a_{ij})_{i,j=1,\ldots,n}$ is an arbitrary $n \times n$ square matrix and

$$|a_{ii}| > \sum_{j,j\neq i} |a_{ij}| \qquad (i = 1, 2, \ldots, n),$$

then A is non-singular. Or equivalently: every eigenvalue of A lies in at least one of the circles

$$|\lambda - a_{ii}| \leqslant \sum_{j,j\neq i} |a_{ij}| \qquad (i = 1, \ldots n)$$

in the complex plane. Theorem 2 of [5] constitutes a special case of our theorem if one put

$$\Phi(t) = \frac{|t|^p}{p}, \qquad 1 < p < \infty.$$

2. In this section we are explaining notations and terms which have been used in formulation of the theorem.

A continuous and convex function $\Phi\colon [0, \infty] \to [0, \infty]$ is said to be an *Orlicz function* if it satisfies the following conditions:

$$\lim_{t\to 0} \big(\Phi(t)/t\big) = 0 \quad \text{and} \quad \lim_{t\to\infty} \big(\Phi(t)/t\big) = \infty.$$

To any such function Φ, there corresponds a function Ψ, called a *complementary function to Φ in the sense of Young*, and defined by means of the formula

$$\Psi(s) = \max_{t\geqslant 0} [ts - \Phi(t)], \qquad s \geqslant 0.$$

Ψ is also the Orlicz function, and Φ is complementary to Ψ in the sense of Young. The *sequence Orlicz space l_Φ* is a set of all sequences $\boldsymbol{x} = \{x_i\}$ of complex numbers with

(O) $$\|\boldsymbol{x}\|_\Phi \overset{\text{df}}{=} \sup \Big|\sum_{i=1}^{\infty} x_i y_i\Big| < \infty,$$

where the supremum is taken over the set of sequences $\boldsymbol{y} = \{y_i\}$ satisfying the condition

$$\sum_{i=1}^{\infty} \Psi(|y_i|) \leqslant 1.$$

l_Φ with the norm (O) (called in the sequel the *Orlicz norm*) is a Banach space. In the same space we also can introduce another norm, equivalent to that of Orlicz:

$$\text{(L)} \qquad \|\boldsymbol{x}\|_{(\Phi)} = \inf\Big\{k > 0 \colon \sum_{i=1}^{\infty} \Phi(|x_i|/k) \leqslant 1\Big\}.$$

It is called the *Luxemburg norm*. Moreover, for every $\boldsymbol{x} \in l_\Phi$, $\|\boldsymbol{x}\|_{(\Phi)} \leqslant \|\boldsymbol{x}\|_\Phi$. Let now $\boldsymbol{x} = \{x_i\} \in l_\Phi$ and $\boldsymbol{y} = \{y_i\} \in l_\Psi$ (Φ complementary to Ψ). Then

$$\Big|\sum_{i=1}^{\infty} x_i y_i\Big| \leqslant \|\boldsymbol{x}\|_\Phi \|\boldsymbol{y}\|_{(\Psi)}.$$

This formula is known as the *strong Hölder's inequality*, and will be of our interest in the sequel.

For the theory of Orlicz spaces we refer to the monograph [6]. The lacking proofs of the facts given here can be rewritten from [1] or [6] almost without changes.

3. A proof of the theorem. Suppose a contrario, that there exists a point $\boldsymbol{x} = \{x_i\}_{i=1,2,\ldots} \in l_\Psi$ such that $\boldsymbol{x} \neq 0$ and $A\boldsymbol{x} = 0$. Without loss of generality we may assume

$$\sum_{i=1}^{\infty} \Psi(|x_i|) = 1.$$

This assumption implies $\|\boldsymbol{x}\|_{(\Psi)} = 1$ where $\| \; \|_{(\Psi)}$ denotes the Luxemburg norm in sequence Orlicz space l_Ψ (see [6], 9.22). Hence, we get by the definition of an operator A that the equations

$$\sum_{j=1}^{\infty} a_{ij} x_j = 0 \qquad (i = 1, 2, \ldots)$$

hold and thus

$$x_i = -\sum_{\substack{j=1 \\ j\neq i}}^{\infty} \frac{a_{ij}}{a_{ii}} x_j \qquad \text{for every} \qquad i = 1, 2, \ldots$$

Therefore the strong Hölder's inequality (see sec. 2) gives the following evaluation:

$$|x_i| \leqslant \left\| \left\{\frac{a_{ij}}{a_{ii}}\right\}_{j=1,2,\ldots} - \mathbf{1}_i \right\|_\Phi \| \, \{x_j\}_{j=1,2,\ldots;\, j\neq i} \, \|_{(\Psi)} .$$

Obviously

$$\|\{x_j\}_{j;j\neq i}\|_{(\Psi)} \leqslant \|\{x_j\}_{j=1,2,\ldots}\|_{(\Psi)} = 1$$

and by virtue of properties of Orlicz function we get

$$\Psi(|x_i|) \leqslant \|\{x_j\}_{j;j\neq i}\|_{(\Psi)} \Psi\left(\left\|\left\{\frac{a_{ij}}{a_{ii}}\right\}_{j=1,2,\ldots} - \mathbf{1}_i\right\|_{\Phi}\right).$$

Now

$$\sum_{i=1}^{\infty} \Psi\,(|x_i|) < \|\{x_j\}_{j=1,2,\ldots}\|_{(\Psi)} \sum_{i=1}^{\infty} \Psi\left(\left\|\left\{\frac{a_{ij}}{a_{ii}}\right\}_{j=1,2,\ldots} - \mathbf{1}_i\right\|_{\Phi}\right) \leqslant 1.$$

This contradicts the assumption and proves the theorem.

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INSTITUTE OF MATHEMATICS OF THE WROCŁAW UNIVERSITY
INSTITUTE OF MATHEMATICS OF THE POLISH ACADEMY OF SCIENCES

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