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A CONJUGACY THEOREM FOR SUBGROUPS OF GL_n CONTAINING THE GROUP OF DIAGONAL MATRICES

BY

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Let Λ be an associative ring with 1, $G = \mathrm{GL}(n, \Lambda)$ be the general linear group of degree n over Λ, and let $D = D(n, \Lambda)$ be its subgroup of diagonal matrices. For a semilocal ring Λ in [2], [4], and [7] a description of subgroups of G containing D was obtained. In this paper, for a *matrix local ring*, i.e., a ring Λ whose factor ring modulo the Jacobson radical J is simple artinian, we find out when two such subgroups are conjugated. Namely, we prove the following result which was announced without proof in [8]:

THEOREM. *Let Λ be a matrix local ring such that Λ/J is isomorphic neither to one of finite fields F_q, with $q \leqslant 5$, nor to the full matrix ring $M(2, F_2)$ of degree 2 over the field of 2 elements. Then if two subgroups F and H of the general linear group $G = \mathrm{GL}(n, \Lambda)$ containing the group $D = D(n, \Lambda)$ of diagonal matrices are conjugated in G, then they are conjugated by a monomial matrix. More precisely, if $xFx^{-1} = H$ for some $x \in G$, then $x = y\pi$, where π is a monomial and $y \in H$.*

Let us remind (see, e.g., [5]) that a subgroup H of a group G is called *pronormal* if for every $x \in G$ the subgroups H and xHx^{-1} are conjugated in the subgroup $\langle H, xHx^{-1}\rangle$ generated by them. Our theorem implies that $D(n, \Lambda)$ is pronormal in $\mathrm{GL}(n, \Lambda)$.

Proof. We will divide the proof into several steps.

1° Let us first introduce necessary definitions and notation. A matrix $\sigma = (\sigma_{ij})$, $1 \leqslant i, j \leqslant n$, of two-sided ideals σ_{ij} of Λ is called a *net* if the relation $\sigma_{ir}\sigma_{rj} \subseteq \sigma_{ij}$ holds for all $i, j, r = 1, \ldots, n$, and a *D-net* if moreover $\sigma_{ii} = \Lambda$ $(1 \leqslant i \leqslant n)$ (see [1], [2], and [4]). To any net σ there corresponds a subring $M(\sigma)$ of the full matrix ring $M(n, \Lambda)$ of degree n over Λ consisting of matrices $a = (a_{ij})$ such that $a_{ij} \in \sigma_{ij}$ for all $i, j = 1, \ldots, n$. The greatest subgroup of $G = \mathrm{GL}(n, \Lambda)$ contained in the multiplicative semigroup $e + M(\sigma)$, where e is the identity matrix, will be called the *net subgroup corresponding to the net* σ and denoted by $G(\sigma)$. For a semilocal ring we have $G(\sigma) = G \cap (e + M(\sigma))$ (see [4], Theorem 1). Let $N(\sigma)$ be the normalizer of $G(\sigma)$ in G.

2° Our first statement is that for a semilocal ring Λ such that neither F_q ($q \leqslant 5$) nor $M(2, F_2)$ appear as direct summands in the decomposition of Λ/J into the direct sum of simple artinian rings (and, in particular, under conditions of the Theorem) for every subgroup H of G containing D there exists a unique D-net σ such that

$$G(\sigma) \leqslant H \leqslant N(\sigma)$$

holds. This is a slight generalisation of Theorem 1 in [7] which in turn is a slight generalisation of Theorem 4 in [4]. The proof of this statement in full details will appear in the "Vestnik of the Leningrad University".

3° In view of 2° we have only to prove that if σ, τ are two D-nets over a matrix local ring Λ, and F, H are two subgroups of G such that $G(\sigma) \leqslant F \leqslant N(\sigma)$ and $G(\tau) \leqslant H \leqslant N(\tau)$, then if $xFx^{-1} = H$ for some $x \in G$, then $x = y\pi$, where π is a monomial and $y \in G(\tau)$. We shall prove this actually when Λ/J is not isomorphic to F_2 or F_3.

4° For net subgroups this conjugacy theorem is known. Namely, from Theorem 6 in [4] and Theorem 1 in [3] it follows that if Λ/J is distinct from F_2, then if $xG(\sigma)x^{-1} = G(\tau)$ for some $x \in G$, then $x = y\pi$ with a monomial π and $y \in G(\tau)$. Of course, for simple artinian rings it is a very particular case of the general theorem on groups with normal root data (see [6]).

In fact, in Section 6 of [4] a little bit more precise result is proved of which a conjugacy criterion for net subgroups is a mere consequence. Let Λ be a simple artinian ring distinct from F_2. If $xDx^{-1} \leqslant G(\sigma)$ for some D-net σ and some $x \in G$, then $x = y\pi$, where $y \in G(\sigma)$ and π is a monomial (see also Proposition 2.14 of [6]).

5° Let us suppose first that Λ is simple artinian. We begin our study with irreducible subgroups. From 4° it follows that for $\Lambda \not\cong F_2$ any subgroup H, $G(\sigma) \leqslant H \leqslant N(\sigma)$, is obtained from $G(\sigma)$ by adding some monomial matrices. Let us call a subgroup H of $G = \mathrm{GL}(n, \Lambda)$, where $\Lambda = M(m, T)$, T being a skew field, *irreducible* if it is not contained in any proper net subgroup of G. Of course, this concept of irreducibility differs from irreducibility of H as a subgroup of $\mathrm{GL}(nm, T)$, but for subgroups containing $D(n, \Lambda)$ these two notions do coincide. It is readily seen that if H is an irreducible subgroup of $G = \mathrm{GL}(n, \Lambda)$ such that $G(\sigma) \leqslant H \leqslant N(\sigma)$ for some D-net σ and $\Lambda \not\cong F_2$, then up to conjugacy with a monomial matrix $G(\sigma) = D(q, M(n/q, \Lambda))$ for some $q|n$, and thus

$$D(q, M(n/q, \Lambda)) \leqslant H \leqslant N(q, M(n/q, \Lambda)),$$

where $N(n, \Lambda)$ is as usual the group of monomial matrices of degree n over Λ.

6° Let us check the following easy fact:

Let $\Lambda = M(m, T)$ be a simple artinian ring distinct from F_2, F_3 and

$a, b \in \Lambda \setminus \{0\}$. Then there exist at least two distinct units $\varepsilon_1, \varepsilon_2 \in \Lambda^*$ such that $a(\varepsilon_i - 1)b \neq 0$.

Let a have a non-zero entry in the column with index p and let b have a non-zero entry in the row with index q. Then $a\lambda e_{pq} b \neq 0$ for any $\lambda \in T^*$. Now $\lambda e_{pq} = (e + \lambda e_{pq}) - e$, where $e + \lambda e_{pq}$ is invertible when $p \neq q$ or $p = q$ and $\lambda \neq -1$. Thus for $T \not\cong F_2, F_3$ we obtain the desired ε's by varying λ.

Let now $T \cong F_3$. If the matrix a has a non-zero column with index p and b has a non-zero row with index $q \neq p$, then the previous argument works. Thus it remains only the case where a has a unique non-zero column and b has a unique non-zero row, namely those with index p. Then matrices $\varepsilon_1 = e + e_{pp}$ and $\varepsilon_2 = e + e_{pp} + e_{qq}$, where $q \neq p$, are precisely what we want.

Thus only the case $T \cong F_2$ remains. If there are three pairwise different indices p, q, r such that either a has non-zero columns with indices p and q and b has a non-zero row with index r, or a has a non-zero column with index p and b has non-zero rows with indices q and r, then the previous argument works. Therefore, we may suppose that a has at most two non-zero columns and b has at most two non-zero rows, namely those with indices p and q. Thus the problem is reduced to the case of $a, b \in M(2, F_2)$. If either of elements a, b is invertible, then it is obvious that such ε's do exist. If now neither of a, b is invertible, then multiplying b from the right and a from the left by invertible matrices and, moreover, b from the left by some invertible matrix and a from the right by the inverse of this matrix, we may suppose that $b = e_{11}$ and $a = e_{11}, e_{12}$ or $e_{11} + e_{12}$. In all these cases it is easy to check directly that the desired ε's do exist.

7° In view of 5° the case of irreducible subgroups over simple artinian rings (possibly considering $M(n/q, \Lambda)$ for some $q|n$ in place of Λ) is solved by the following statement:

Let Λ be a simple artinian ring distinct from F_2 and F_3. Then if $xD(n, \Lambda)x^{-1} \leqslant N(n, \Lambda)$ for some $x \in GL(n, \Lambda)$, then $x \in N(n, \Lambda)$.

Let x be a non-monomial matrix. This means that it has at least two non-zero entries in some column, say in the r-th one. Let these entries be x_{pr} and x_{qr}, $q \neq p$. The inverse $x^{-1} = (x'_{ij})$ has at least one non-zero entry in the r-th row, say $x'_{rs} \neq 0$. Now take

$$y = x\big(e + (\varepsilon - 1)e_{rr}\big)x^{-1} = e + \sum_{i,j} x_{ir}(\varepsilon - 1)x'_{rj}e_{ij}$$

for $\varepsilon \in \Lambda^*$. At least one of the indices p, q is distinct from s, say p. Then let us look at the entries of y in positions (p, s) and (s, s). If either of elements x_{sr}, x'_{rs} is not invertible, then

$$y_{ss} = 1 + x_{sr}(\varepsilon - 1)x'_{rs} \neq 0 \quad \text{for any } \varepsilon \in \Lambda^*,$$

and if both $x_{sr}, x'_{rs} \in \Lambda^*$, then only one value of ε, namely $1 - (x'_{rs}x_{sr})^{-1}$, is prohibited. Now by 6° we may choose such an ε subject to the condition

$$y_{ps} = x_{pr}(\varepsilon - 1)x'_{rs} \neq 0.$$

Thus the matrix y is not monomial since it has at least two non-zero elements in the s-th column.

8° To complete the proof of the Theorem for simple artinian rings we need the following fact. Let σ be a D-net of degree n over a simple artinian ring Λ and let $\nu = (n_1, \ldots, n_t)$ be a partition of n. We may write σ in the block form $\sigma = (\sigma^{kl})$, where σ^{kl} is a matrix of ideals having n_k rows and n_l columns. We shall say that σ is a *block triangular net of type* ν if every block σ^{kk} consists only of unit ideals and every block σ^{kl}, $k > l$, consists only of zero ideals. Theorem 4 of [2] states that every D-net σ over a simple ring is similar to a block triangular D-net, that is, there exists a permutation $\pi \in S_n$ such that the net $\sigma^{\pi} = (\sigma_{\pi i, \pi j})$ is block triangular with respect to some partition $\nu = (n_1, \ldots, n_t)$ and the summands $n_1, \ldots, n_t$ are uniquely determined up to their order.

Let σ be a block triangular D-net of type $\nu = (n_1, \ldots, n_t)$ and $G(\sigma)$ the corresponding net subgroup. For every $k = 1, \ldots, t$ there is a natural projection φ_k of $G(\sigma)$ on $\mathrm{GL}(n_k, \Lambda)$, assigning to each matrix $a \in G(\sigma)$ its block a^{kk}.

9° Theorem 1 of [1] says that if a ring Λ is generated by its units and there exists a unit $\varepsilon \in \Lambda^*$ such that $\varepsilon - 1$ is also a unit, then the group $D = D(n, \Lambda)$ is abnormal in the group $B = B(n, \Lambda)$ of upper triangular matrices, i.e., every $b \in B$ belongs to the subgroup $\langle D, bDb^{-1} \rangle$ generated by D and bDb^{-1}.

10° Now we may complete the proof of the Theorem for simple artinian rings. Set $G(\sigma) \leqslant F \leqslant N(\sigma)$, $G(\tau) \leqslant H \leqslant N(\tau)$ and $xFx^{-1} = H$ for some $x \in G$. There exists a unique smallest D-net ω such that $H \leqslant G(\omega)$. Therefore, xFx^{-1}, and hence xDx^{-1} is contained in $G(\omega)$, and thus by 4° we have $x = y\pi_1$ for some $y \in G(\omega)$ and some permutation matrix π_1. Set $F_1 = \pi_1 F \pi_1^{-1}$. Then $yF_1 y^{-1} = H$ with $y \in G(\omega)$, and thus $F_1 \leqslant G(\omega)$. Conjugating with a monomial matrix we may suppose in view of 8° that ω has a block triangular form with respect to some partition $\nu = (n_1, \ldots, n_t)$. Now we shall regard all other objects, as the matrix y, nets σ^{π_1} and τ, etc., as block-triangular with respect to this partition. For every $s = 1, \ldots, t$ we get

$$y^{ss} \varphi_s(F_1)(y^{ss})^{-1} = \varphi_s(H).$$

The group $\varphi_s(H)$ is irreducible (else $G(\omega)$ would not be the smallest net subgroup containing H). Thus $\varphi_s(F_1)$ is also irreducible and by 7° we get $y^{ss} = a^{ss}\pi_2^s$, where $a^{ss} \in G(\tau^{ss})$ and π_2^s is a permutation matrix of degree n_s. Now set

$$a = \mathrm{diag}(a^{11}, \ldots, a^{tt}) \quad \text{and} \quad \pi_2 = \mathrm{diag}(\pi_2^1, \ldots, \pi_2^t).$$

Then $a^{-1}y = b\pi_2$, where $b \in G(\omega)$ is such that $b^{ss} = e$ for every $s = 1, \ldots, t$. Now $bDb^{-1} \leqslant H$ and the matrix b is upper triangular. By 9° we get

$$b \in \langle D, bDb^{-1} \rangle \leqslant G(\tau)$$

and the proof of the Theorem for simple artinian rings is completed since $x = (ab)(\pi_2\pi_1)$, where $ab \in G(\pi)$ and $\pi_2\pi_1$ is a permutation matrix.

11° Let now Λ be a matrix local ring, and J its Jacobson radical. For any net $\sigma = (\sigma_{ij})$ there correspond a net $\sigma^* = (\sigma_{ij}+J)$ over Λ and a net $\bar{\sigma} = ((\sigma_{ij}+J)/J)$ over $\bar{\Lambda} = \Lambda/J$. Let now $G(\sigma) \leqslant F \leqslant N(\sigma)$, $G(\tau) \leqslant H \leqslant N(\tau)$ and $xFx^{-1} = H$ for some $x \in G = \mathrm{GL}(n, \Lambda)$. Denote by G_J the principal congruence subgroup modulo J, that is, the group of all matrices congruent to e modulo J. Then for $\bar{F} = FG_J/G_J$ and $\bar{H} = HG_J/G_J$ the following conditions are satisfied: $G(\bar{\sigma}) \leqslant \bar{F} \leqslant N(\bar{\sigma})$, $G(\bar{\tau}) \leqslant \bar{H} \leqslant N(\bar{\tau})$, and $\bar{x}\bar{F}\bar{x}^{-1} = \bar{H}$, where $\bar{x}$ is the image of $x \in G$ with respect to the natural projection $G \to \bar{G} = \mathrm{GL}(n, \bar{\Lambda})$. Now the ring $\bar{\Lambda}$ is simple artinian and we have just proved that this implies that $\bar{x} = \bar{y}\pi$ for some $y \in G(\tau)$ and some permutation matrix π. In particular, $\bar{x}G(\bar{\sigma})\bar{x}^{-1} = G(\bar{\tau})$. Thus $xG(\sigma)x^{-1} \leqslant G(\tau^*)$.

We have also inclusions $xG(\sigma)x^{-1} \leqslant xFx^{-1} = H \leqslant N(\tau)$, and thus $xG(\sigma)x^{-1}$ is contained in the intersection of $G(\tau^*)$ and $N(\tau)$. But by Lemma 8 of [3] this intersection equals $G(\tau)$. Thus $xG(\sigma)x^{-1} \leqslant G(\tau)$. But starting with $x^{-1}Hx = F$ we could in the same manner prove that $x^{-1}G(\tau)x \leqslant G(\sigma)$. Thus $xG(\sigma)x^{-1} = G(\tau)$. Now by 4° this implies that $x = y\pi$, where $y \in G(\tau)$ and π is a permutation matrix. The proof of our Theorem is now complete.

Remarks. (a) For the fields F_2 and F_3 the assertion of the Theorem is actually false. In the case of F_2 one should look at the matrices $e+e_{12}$ and $e_{12}+e_{21}$ in $\mathrm{GL}(2, F_2)$. In the case of F_3 we put $c = e_{11}+e_{12}+e_{21}-e_{22}$. Then for $D = D(2, F_3)$ we have

$$cDc^{-1} = \{\pm e, \pm(e_{12}+e_{21})\}.$$

Thus D and cDc^{-1} generate the group $N = N(2, F_3)$ of monomial matrices, but they are obviously not conjugated in N.

(b) It is worth mentioning that in the case of infinite fields the Theorem is an immediate consequence of 4°. In fact, $G(\sigma)$'s may be characterised as the only Zariski-connected subgroups of G containing D (all such subgroups are algebraic). Thus if two subgroups F and H containing D are conjugated by $x \in G$, then $xF^0x^{-1} = H^0$, where F^0 and H^0 are the connected components of e in F and H, respectively. But F^0 and H^0 are of the forms $G(\sigma)$ and $G(\tau)$ for some D-nets σ and τ and we may apply 4°.

(c) The Theorem is no more valid for general semilocal rings. Of course, it follows immediately that D is pronormal in G for any ring which is the direct sum of matrix local rings satisfying the conditions of the Theorem. But for any ring not in this class this is false. Let Λ be an indecomposable ring other than a matrix local one. If there exists a unit $\varepsilon \in \Lambda^*$ such that $\varepsilon - 1 \in \Lambda^*$, then $N/D \cong S_n$, but

$$N_G(DG_J)/DG_J \cong S_n \times \ldots \times S_n,$$

where the number of S_n's equals the number of summands in the decomposition of Λ/J into the direct sum of simple artinian rings. Thus D cannot be pronormal. It is even easier to give such counterexamples when there does not exist such a unit ε.

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