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A JACOBSON-SEMISIMPLE BANACH ALGEBRA WITH A DENSE NIL SUBALGEBRA

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In this note we give an example of an algebra as described in the title. This answers a question of Grabiner [1].

Let A be the free Banach algebra on generators X_n ($n \in \mathbf{Z}^+$) of norm 1, with the relations: $M = 0$ for every monomial $M = X_{i_1} \ldots X_{i_r}$ containing more than n occurrences of X_n, where $n = \max\{i_1, \ldots, i_r\}$. (The Banach algebra A is obtained by taking the algebra A_0 generated algebraically by $\{X_n \colon n \in \mathbf{Z}^+\}$ with these relations, giving it the norm

$$\Big\| \sum_{i=0}^{n} \lambda_i M_i \Big\| = \sum_{i=0}^{n} |\lambda_i|$$

(λ_i scalars, M_i monomials), and completing.)

The dense subalgebra A_0 (the algebra of polynomials) is clearly nil, since if $P \in A_0$, we can find

$$N = \max\{n \colon X_n \text{ occurs in } P\}$$

and then $P^{(N+1)!} = 0$.

We show that A is Jacobson-semisimple by proving that it has no non-zero ideals of topologically nilpotent elements (see [2], Chapter II, Section 3). We show that, given a non-zero $T \in A$, there is a $Y \in A$ with TY not topologically nilpotent. Now T is a linear combination of monomials M_i,

$$T = \sum_{i=0}^{\infty} \lambda_i M_i \quad \text{with} \quad \|T\| = \sum_{i=0}^{\infty} |\lambda_i| < \infty \text{ and } \lambda_0 \neq 0.$$

Let $N > \max\{n \colon X_n \text{ occurs in } M_0\}$, and write

$$Y = \sum_{i=0}^{\infty} 2^{-i} X_{N+i}.$$

We consider $(TY)^n$. This will be a linear combination of monomials; each monomial being of the form

$$M_{k_1} X_{N+j_1} M_{k_2} X_{N+j_2} \ldots M_{k_n} X_{N+j_n}, \tag{1}$$

where $j_1, \ldots, j_n, k_1, \ldots, k_n \in \{0, 1, 2, 3, \ldots\}$. We fix attention on those monomials of the form

$$M_0 X_{N+j_1} M_0 X_{N+j_2} \ldots M_0 X_{N+j_n} \tag{2}$$

and we are worried lest, in computing $(TY)^n$, cancellation should occur between these and other monomials of form (1). However, if, in (1), any of the M_k should contain an X_r with $r \geqslant N$, then the total number of such X_r in (1) would exceed n, and so (1) could not equal (2), where the total number of such X_r is precisely n. If, on the other hand,

$$N > \max\{m\colon X_m \text{ occurs in } M_k\}$$

for every M_k occurring in (1), then (1) and (2) can only be equal if $M_{k_i} = M_0$ $(1 \leqslant i \leqslant n)$. Thus monomials (2) are distinct from all other monomials in $(TY)^n$; and, of course, different sequences $(j_1, \ldots, j_n)$ give distinct monomials (2). Thus $\|(TY)^n\|$ is not less than the modulus of the coefficient of one of the monomials of form (2), provided that the sequence $(j_1, \ldots, j_n)$ be chosen so that (2) does not vanish. Such a sequence is given by

$$j_r = \max\{i\colon N^i \text{ divides } r\},$$

and this yields

$$\|(TY)^n\| \geqslant 2^{-t} |\lambda_0|^n,$$

where, if

$$n = a_s N^s + \ldots + a_1 N + a_0 \qquad (0 \leqslant a_0, a_1, \ldots, a_s < N),$$

then

$$t = \frac{a_s(N^s - 1) + \ldots + a_1(N-1)}{N-1} \leqslant \frac{n}{N-1}.$$

Thus

$$\|(TY)^n\|^{1/n} \geqslant 2^{-1/(N-1)} |\lambda_0|,$$

so TY is not topologically nilpotent. This completes the proof.

REFERENCES

[1] S. Grabiner, *Nilpotents in Banach algebras*, The Journal of the London Mathematical Society 14 (2) (1976), p. 7-12.

[2] C. E. Rickart, *General theory of Banach algebras*, Princeton, N. J., 1960.

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