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SEQUENTIAL SPACE METHODS IN GENERAL TOPOLOGICAL SPACES

BY

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When a topology is specified by its open sets, the importance of the notions of basis and subbasis needs no comment. But when a topology is specified by its convergence class (as defined by Kelley [5], p. 74), no comparable general theory seems to have been developped. In this paper* and [9] and [10] such a development is begun.

The study of one important special case, now called sequential spaces, is as old as topology itself. (A *sequential space* can be described as a topological space whose convergence class is determined by its convergent sequences; for recent work on sequential spaces see [2], [3], [4], [11].) By replacing sequences with nets, we get more general machinery which applies to any topological space. This leads in Section 1 to natural higher cardinality generalizations of the notions of sequential and Fréchet spaces. Franklin's characterization [3] of sequential spaces as quotients of first countable spaces is here generalized to arbitrary topological spaces (Theorem 3.3).

There are several advantages to be gained from this development. First, for a given point in a space the cardinality of a convergence basis (defined in Section 1) is never larger than the cardinality of an open set basis (Theorem 3.1) and in many cases it is strictly smaller. Second, in passing to quotients a convergence subbasis is preserved (Theorem 2.1), whereas an open set basis is usually not preserved. At the same time this provides a new characterization of the quotient topology.

Examples of topologies which arose from other contexts but which can be studied by the methods of this paper can be found in [8] and [12]. This approach to general topology is also considered in [1] and [14]; there, however, the objectives are somewhat different.

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1. The notion of a convergence subbasis. Let X be a set and $\mathscr{C}$ be a class of net pairs in X (i.e., pairs of the form (Q, x) where Q is a net in X and $x \in X$). The convergence associated with $\mathscr{C}$ can be studied topologically by means of the largest topology on X in which the given nets are topologically convergent. (For topological terminology see [5]; a larger topology is one with more open sets.)

We assume

(i) *If* $(\{x_\nu, \nu \in D\}, x) \in \mathscr{C}$ *and* E *is a cofinal subset of* D, *then* $(\{x_\nu, \nu \in E\}, x) \in \mathscr{C}$.

The closure operator on X can now be defined: For a subset Y of X, $\mathrm{cl}\, Y$ is the smallest set containing Y and closed with respect to the formation of limits of $\mathscr{C}$-nets.

PROPOSITION 1.1. *The closure operator satisfies the Kuratowski axioms and hence defines a topology* t *on* X. *This is the largest topology on* X *in which* $\mathscr{C}$*-nets converge. It is a* T_1 *topology if and only if* $\mathscr{C}$ *also satisfies*:

(ii) *If* $x_\nu = x$ *for each* $\nu \in D$ *and* $y \neq x$, *then* $(\{x_\nu, \nu \in D\}, y) \notin \mathscr{C}$.

Proof. The only Kuratowski axiom which requires comment is: $\mathrm{cl}(A \cup B) = \mathrm{cl}\, A \cup \mathrm{cl}\, B$. Since $A \cup B \subset \mathrm{cl}\, A \cup \mathrm{cl}\, B \subset \mathrm{cl}(A \cup B)$, it suffices to show that $\mathrm{cl}\, A \cup \mathrm{cl}\, B$ is closed. Suppose that $(\{x_\nu, \nu \in D\}, x) \in \mathscr{C}$ with $x_\nu \in \mathrm{cl}\, A \cup \mathrm{cl}\, B$. Either $\{\nu : x_\nu \in \mathrm{cl}\, A\}$ or $\{\nu : x_\nu \in \mathrm{cl}\, B\}$ is cofinal in D; hence, by axiom (i), $x \in \mathrm{cl}\, A \cup \mathrm{cl}\, B$, and this set is closed. The other statements clearly follow.

There is another closure operator, the generalization of the sequential closure, which plays a secondary role here. (In Čech [1], it has primary importance.) Define the *$\mathscr{C}$-closure* of a subset Y to be the set formed by adjoining to Y the limits of those $\mathscr{C}$-nets which lie entirely in Y. Clearly, $\mathscr{C}$-cl Y is a subset of t-cl Y, and it can be a proper subset. However, t-cl Y can be constructed inductively by iteration of the $\mathscr{C}$-closure operator.

We now define a notion of Baire order which generalizes the classical definition. For each z in t-cl Y there is a smallest ordinal η such that z is in the η-th iterate of the $\mathscr{C}$-closure of Y. Then z is said to have *Baire order* η (write ord $z = \eta$) with respect to $\mathscr{C}$ and Y. It is also convenient to define Baire order of Y (ord Y) as $\sup\{\mathrm{ord}\, z \colon z \in t\text{-cl}\, Y\}$. (The classical definitions of Baire order can be recovered for example by letting X be the set of all bounded real-valued functions on a topological space E, Y be the set of all continuous such functions, and $\mathscr{C}$ be either the pointwise or the pointwise monotone convergent sequence pairs; see [6].)

In view of the order inverting one-to-one correspondence between the topologies on X and the convergence classes on X([5], p. 76), the topology t is the topology with the smallest convergence class containing $\mathscr{C}$. Thus $\mathscr{C}$ will be called a *convergence subbasis* for t. If $\mathscr{C}$ satisfies the further condition that every subset has Baire order $\leqslant 1$ (i.e., the $\mathscr{C}$-closure operator is idempotent and coincides with the t-closure operator), then $\mathscr{C}$ will be called a *convergence basis* for t. Note that a convergence basis for t can still be a proper subset of the convergence class of t.

Let $\mathfrak{m}$ be an infinite cardinal number. A topological space which has a convergence subbasis in which all of the nets are $\mathfrak{m}$-nets (i.e., the cardinality of each directed set is $\leqslant \mathfrak{m}$) is called an $\mathfrak{m}$-*sequential space.* A space which has a convergence basis consisting of $\mathfrak{m}$-nets is called an $\mathfrak{m}$-*Fréchet space.* A sequential space in the usual sense can be described as a space which has a convergence subbasis consisting of sequences; a Fréchet space as one which has a convergence basis of sequences. Thus, for the case $\mathfrak{m} = \aleph_0$ these definitions are equivalent to the usual ones, since every non-trivial $\aleph_0$-net has a cofinal sequence.

It will be shown below (Theorem 3.1) that any topological space can, for sufficiently large $\mathfrak{m}$, be so described. Nevertheless, we continue to state results in terms of a convergence subbasis $\mathscr{C}$, since in the applications this greater generality is needed. (Even in the ordinary sequential space case, it is often convenient to work with a convergence subbasis which is a proper subset of the set of all convergent sequence pairs; see for example [7].) Of course, using a smaller convergence subbasis may increase Baire order. However, for $\mathfrak{m}$-sequential spaces there is an upper bound. This result generalizes the fact that every Baire function in the classical sense has countable Baire order.

PROPOSITION 1.2. *If X is $\mathfrak{m}$-sequential with any convergence subbasis, then no element of X has Baire order equal to the least ordinal of cardinality $\mathfrak{m}^+$. ($\mathfrak{m}^+$ denotes the cardinal successor of $\mathfrak{m}$.)*

Proof. Let ω_a denote the least ordinal of cardinality $\mathfrak{m}^+$. Since a is of the form $\beta+1$ for some β, it follows that ω_a is regular. Thus ω_a is not the supremum of any set S of strictly smaller ordinals if $\operatorname{card} S \leqslant \mathfrak{m}$. The result follows.

The next result extends results of Dudley [2], p. 488-489, and Franklin [3], Proposition 7.2.

PROPOSITION 1.3. *Let $\mathscr{C}$ be a convergence subbasis for a topology t on X, let Y be a subset of X, and let $\mathscr{D}$ be the trace of $\mathscr{C}$ on Y (i.e., $\mathscr{D} = \{(\{x_\nu, \nu \in D\}, x) \in \mathscr{C}: x_\nu \in Y \text{ for each } \nu \text{ and } x \in Y\}$). Then $\mathscr{D}$ is a convergence subbasis for a topology on Y which is in general larger than the relative topology. This induced topology coincides with the relative topology on Y if Y is closed or open in X. The two topologies coincide for all subsets of X without restriction if and only if $\mathscr{C}$ is in fact a convergence basis.*

Proof. $\mathscr{D}$ is a convergence subbasis for some topology on Y; call it u. Since u is the largest topology on Y in which these nets converge, u is larger than the relative topology.

To show that these topologies on Y coincide when Y is closed or open, or when $\mathscr{C}$ is a convergence basis, we must show that, for a subset Z of Y, $u\text{-cl}\, Z \supset (t\text{-cl}Z) \cap Y$. (The opposite inclusion is true in general.) Assume x is in $(t\text{-cl}Z) \cap Y$ with $\operatorname{ord} x = \lambda$ with respect to $\mathscr{C}$, and proceed by induction on λ. If λ is 0 or 1, then x is clearly in $u\text{-cl}Z$, and the proof is complete for the case in which $\mathscr{C}$ is a convergence basis. For $\lambda > 1$ we must consider the cases Y is closed and Y is open; the details are given here for the closed case only. By the inductive hypothesis, there exists $(\{x_\nu, \nu \in D\}, x) \in \mathscr{C}$ with $x_\nu \in t\text{-cl}Z$ and $\operatorname{ord} x_\nu < \lambda$. Since Y is closed, $x_\nu \in Y$. Thus $(\{x_\nu, \nu \in D\}, x)$ is in $\mathscr{D}$ and $x \in u\text{-cl}Z$.

It remains to show that these two topologies on Y are the same without restriction on Y only if $\mathscr{C}$ is a convergence basis. We use Dudley's argument: if $\mathscr{C}$ is not a convergence basis on X, there is at least one Baire order 2 situation; i.e., a subset Z of X with $x \in t\text{-cl}Z$ and $\operatorname{ord} x = 2$. Let $Y = Z \cup \{x\}$. Then $x \in (t\text{-cl}Z) \cap Y$, but $x \notin u\text{-cl}Z$, and the topologies are different.

Corollary 1.4. *In an $\mathfrak{m}$-sequential space every open or closed subspace is $\mathfrak{m}$-sequential. A topological space is $\mathfrak{m}$-Fréchet iff it is hereditarily $\mathfrak{m}$-Fréchet iff it is hereditarily $\mathfrak{m}$-sequential.*

2. Quotient spaces. The first part of the following theorem shows that the usual criterion for continuity in terms of nets in the domain space can be simplified if a convergence subbasis is present: it suffices to look at subbasic nets. Related results are found in [2], Theorem 2.2, and [13], p. 529. Let $f\mathscr{C}$ denote the set of all net pairs $(\{fx_\nu, \nu \in D\}, fx)$ for $(\{x_\nu, \nu \in D\}, x)$ in $\mathscr{C}$.

Theorem 2.1. *Let X and Z be arbitrary topological spaces, with f a function on X to Z, and let $\mathscr{C}$ be a convergence subbasis for X.*

(a) *The function f is continuous if and only if $f\mathscr{C}$ is contained in the convergence class of Z.*

(b) *If f is surjective, then the topology of Z is the quotient topology if and only if $f\mathscr{C}$ is a convergence subbasis for Z.*

Proof. (a) The condition is clearly necessary. To show that it is sufficient, let A be a closed subset of Z and show $f^{-1}(A)$ is closed in X. Suppose $(\{x_\nu, \nu \in D\}, x) \in \mathscr{C}$ with $x_\nu \in f^{-1}(A)$ for each ν. Since A is closed and by hypothesis $\{fx_\nu, \nu \in D\}$ converges to fx, we have $fx \in A$. Thus $x \in f^{-1}(A)$ and the proof is complete.

(b) The quotient topology for Z is the largest topology (smallest convergence class) such that f is continuous. But the smallest convergence class containing $f\mathscr{C}$ is that for which $f\mathscr{C}$ is a convergence subbasis. (See Proposition 1.1 and remarks following it.)

As a generalization of [3], Proposition 1.2, we now have

Corollary 2.2. *Every quotient of an $\mathfrak{m}$-sequential space is $\mathfrak{m}$-sequential.*

Remarks 2.3. If $\mathscr{C}$ is a convergence basis, or even the entire convergence class of X, we can still only conclude in general that $f\mathscr{C}$ is a convergence subbasis for the quotient space. (See for example [3], Proposition 1.12.) This can be stated in another way: Not every convergent net in the quotient space "lifts" to a convergent net in X; however, those nets which do "lift" are adequate to describe the quotient topology (i.e., they form a convergence subbasis). Thus even in applications where there seems to be no natural convergence subbasis for X, there is always a natural convergence subbasis in the quotient.

3. $\mathfrak{m}$-sequential spaces. The main result of this section (Theorem 3.3) is a characterization of $\mathfrak{m}$-sequential spaces which generalizes Franklin's characterization of sequential spaces.

THEOREM 3.1. *If (X, t) is a topological space and $\mathfrak{m}$ is an infinite cardinal number, then each of the following conditions implies the next:*

(a) *(X, t) has local character $\leqslant \mathfrak{m}$ (i.e., each point has a neighborhood base of cardinality $\leqslant \mathfrak{m}$).*

(b) *(X, t) is $\mathfrak{m}$-Fréchet.*

(c) *(X, t) is $\mathfrak{m}$-sequential.*

(d) *(X, t) is determined by sets of cardinality $\leqslant \mathfrak{m}$ in the following sense: For any subset A of X, each point in t-$\mathrm{cl}A$ is in t-$\mathrm{cl}B$ for some subset B of A with $\mathrm{card}\,B \leqslant \mathfrak{m}$.*

(e) *(X, t) is $\exp \mathfrak{m}$-Fréchet.*

Proof. (a) $\rightarrow$ (b) This is a generalization of the fact that first countable spaces are Fréchet spaces and a similar proof applies.

(b) $\rightarrow$ (c) is trivial.

(c) $\rightarrow$ (d) For $z \in t$-$\mathrm{cl}A$, the existence of B is established by transfinite induction on the Baire order of z with respect to A.

(d) $\rightarrow$ (e) is clear.

Remarks 3.2. There are well-known examples, for the case $\mathfrak{m} = \aleph_0$, to show that all of the conditions in the theorem are distinct. However, it is proved in [9] that conditions (a), (b) and (c) are equivalent (for arbitrary $\mathfrak{m}$) in topological spaces which are products of ordered spaces (i.e., products in which each coordinate space has the order topology arising from a total order). This equivalence is extended in [10], Section 3, to a larger class of spaces.

THEOREM 3.3. *A topological space is $\mathfrak{m}$-sequential if and only if it is a quotient of a space of local character $\leqslant \mathfrak{m}$.*

Proof. In one direction the result follows from Theorem 3.1 and Corollary 2.2. In the other direction, the construction generalizes that of Franklin [3], Proposition 1.12. Let X be an $\mathfrak{m}$-sequential space with $\mathscr{C}$ a convergence subbasis consisting of $\mathfrak{m}$-nets; we construct a space T of which X is a quotient. For each $(\{x_\nu, \nu \in D\}, x) \in \mathscr{C}$, let $S(\{x_\nu\}, x) = \{x\} \cup \cup$ range $\{x_\nu, \nu \in D\}$ be a topological space in which each x_ν distinct from x is isolated and $\{x_\nu, \nu \in D\} \to x$ (i.e., it has a convergence basis generated by this one net). If x is isolated in X, let $S(x) = \{x\}$. Let T be the disjoint topological sum of all such S. Thus T has a convergence basis formed by taking the union of the convergence bases in each S. The surjection $T \to X$ is continuous and is a quotient map by Theorem 2.1. T has character $\leqslant \mathfrak{m}$ since each summand does. (As a point in $S(\{x_\nu\}, x)$, x has character $\leqslant \mathfrak{m}$ because it has a neighborhood base indexed by D and $\operatorname{card} D \leqslant \mathfrak{m}$.)

Added in proof. Theorem 2.1 shows that the maps which preserve convergence subbases are the quotient maps. F. Siwiec has shown that the maps which preserve convergence bases are precisely the continuous pseudo-open maps.

COROLLARY. *A space is $\mathfrak{m}$-Frechet iff it is a continuous pseudo-open image of a space of local character $\leqslant \mathfrak{m}$.*

This extends known results ([3], Prop. 2.4) even in the countable case by removing the Hausdorff hypothesis.

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