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NONUNIFORM CENTRAL LIMIT BOUNDS AND THEIR APPLICATIONS

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1. Introduction. Let $\mathcal{G}$ be a set of functions $g(x)$, defined on the set of real numbers $\boldsymbol{R}$ and satisfying the following conditions:

$g(x)$ is nonnegative, even, and nondecreasing on $[0, \infty)$,

$x/g(x)$ is defined for all $x \in \boldsymbol{R}$ and nondecreasing on $[0, \infty)$.

We consider a sequence $\{X_k, k \geqslant 1\}$ of independent random variables with $\mathrm{E}X_k = 0$, $\mathrm{E}X_k^2 = \sigma_k^2$, $\mathrm{E}X_k^2 g^s(X_k) < \infty$, $k \geqslant 1$, where $s > 0$ is some fixed number.

Let

$$S_n = \sum_{k=1}^{n} X_k, \quad B_n^2 = \sum_{k=1}^{n} \sigma_k^2, \quad F_n(t) = P[S_n < tB_n],$$

and

$$L_n^s(g) = \Big[\sum_{k=1}^{n} \mathrm{E}[X_k^2 \cdot g^s(X_k)]\Big]/B_n^2 g^s(B_n).$$

The aim of this note is to give some nonuniform rates of convergence to normality. The results obtained strengthen the analogous ones of Bikelis [3], Michel [9], Maejima [8] as well as of Ghosh and Dasgupta [6].

2. Nonuniform central limit bounds. We begin with a lemma, which generalizes Lemma 2 of [10], p. 139.

LEMMA. *Let $X_1, X_2, \ldots, X_n$; $Y_1, Y_2, \ldots, Y_n$ be random variables and let $g_1, g_2, \ldots, g_n$; $f_1, f_2, \ldots, f_n$ be nonnegative measurable functions such that for all $s \geqslant 0$, $r \geqslant 0$ and $k = 1, 2, \ldots, n$,*

$$\mathrm{E}[g_k^s(X_k) f_k^r(Y_k)] < \infty.$$

Then for every $s < s'$ we have

$$L_n^s(g, f) \leqslant [L_n^{s'}(g, f)]^{s/s'},$$

where

$$L_n^s(g, f) = \sum_{k=1}^{n} \mathrm{E}[g_k^r(X_k) f_k^s(Y_k)]/\sum_{k=1}^{n} \mathrm{E}[g_k^r(X_k)].$$

Proof. First, let us note that the function

$$h(s) = \log \sum_{k=1}^{n} \mathrm{E}\,[g_k^r(X_k) f_k^s(Y_k)], \qquad s > 0,$$

is convex from below. Indeed, by Hölder's inequality applied twice, for $s < s'$ we obtain

$$\Big(\sum_{k=1}^{n} \mathrm{E}\,[g_k^r(X_k) f_k^{s'}(Y_k)]\Big)^2 \leqslant \sum_{k=1}^{n} \mathrm{E}\,[g_k^r(X_k) f_k^{s'-s}(Y_k)] \sum_{k=1}^{n} \mathrm{E}\,[g_k^r(X_k) f_k^{s+s'}(Y_k)].$$

Taking logarithms, we have

$$h(s') \leqslant \tfrac{1}{2}\big(h(s'-s)+h(s'+s)\big).$$

Consequently, $h(s)$ is convex from below.

Note that

$$h(0^+) = \log \sum_{k=1}^{n} \mathrm{E}\,[g_k^r(X_k)].$$

Now, we prove that the function $\big(h(s')-h(0^+)\big)/s'$ is nondecreasing for $s' > 0$.

Let us put $p_1 = 1-s/s'$ and $p_2 = s/s'$, where $s < s'$. Since the function $h(s)-h(0^+)$ is convex from below, we have

$$h(p_1 x_1 + p_2 x_2) - h(0^+) \leqslant p_1 [h(x_1)-h(0^+)] + p_2 [h(x_2)-h(0^+)].$$

Putting $x_2 = s'$ and $x_1 \to 0^+$, we obtain

$$h(s') - h(0^+) \geqslant s' [h(s)-h(0^+)]/s.$$

Thus the Lemma is proved.

All theorems in this paper are proved under the same conditions given above; therefore we do not repeat them explicitly in the formulations.

Let us put

$$\Delta_n(t) = |F_n(t) - \Phi(t)|, \qquad t \in \boldsymbol{R},\ n \geqslant 1,$$

where Φ denotes the standard normal distribution function.

THEOREM 1. *There exist constants b, $r > 0$ (depending on s) such that for all $n \geqslant 1$ and every t with*

$$t^2 \geqslant 2s^{-1}(1+s)\log_+\big(1/L_n^s(g)\big)$$

we have

$$\Delta_n(t) \leqslant b t^{-2(2+s)} L_n^s(g) + T_n(r),$$

where

$$T_n(r) = \sum_{k=1}^{n} P\,[|X_k| > r|t| B_n]$$

and

$$\log_+ x = \max\{0, \log x\}.$$

Proof. Without loss of generality we may and do assume that $t > 0$ (if not, replace the random variables X_k by $-X_k$, $1 \leqslant k \leqslant n$). Define

$$X_k^* = X_k \cdot I(|X_k| \leqslant r|t| B_n),\ 1 \leqslant k \leqslant n, \qquad S_n^* = \sum_{k=1}^{n} X_k^*,$$

where $r = [2(1+s)(2+s)]^{-1}$. Then

$$P[S_n \geqslant tB_n] \leqslant P[S_n^* \geqslant tB_n] + T_n(r).$$

Furthermore, by Lemma 2 of [5], p. 166, for

$$t^2 \geqslant 2s^{-1}(1+s)\log_+(1/L_n^s(g)),$$

we get

$$\Phi(-t) \leqslant (2\pi)^{-1/2} t^{-1} \exp(-t^2/2(1+s)) L_n^s(g) \leqslant b_1 t^{-2(2+s)} L_n^s(g),$$

where (and in the sequel) $b, b_1, b_2, \ldots$ denote positive constants which may depend only on s and the same symbols may be used for different constants. Thus

$$\Delta_n(t) \leqslant b_1 t^{-2(2+s)} L_n^s(g) + P[S_n^* \geqslant tB_n] + T_n(r). \tag{2.1}$$

Let

$$h = t^{-1} B_n^{-1} [2\log_+(1/L_n^s(g)) + r^{-1}\log t].$$

Then

$$\begin{aligned} P[S_n^* \geqslant tB_n] &\leqslant \exp(-htB_n)\,\mathrm{E}[\exp(hS_n^*)] \\ &= t^{-1/r}[L_n^s(g)]^2\,\mathrm{E}[\exp(hS_n^*)]. \end{aligned} \tag{2.2}$$

But, by our assumptions, we have

$$\mathrm{E}[\exp(hS_n^*)] = \prod_{k=1}^{n} \mathrm{E}[\exp(hX_k^*)]$$

and for $1 \leqslant k \leqslant n$ one can obtain the following inequalities:

$$h\,\mathrm{E}|X_k^*| \leqslant h\sigma_k^2/rtB_n \leqslant b_2\sigma_k^2/B_n^2, \qquad \mathrm{E}(X_k^*)^2 \leqslant \sigma_k^2,$$

$$h^3\,\mathrm{E}|X_k^*|^3 \exp(hrtB_n) \leqslant h^3 t^2 [L_n^s(g)]^{-2r}\,\mathrm{E}[X_k^2 g^{s^*}(X_k)]\, rB_n/g^{s^*}(rtB_n),$$

where $s^* = \max\{s \geqslant z > 0\colon x/g^z(x) \text{ is nondecreasing}\}$. Putting in our Lemma $Y_k = X_k$, $f_k(x) = x$, $r = 2$, $1 \leqslant k \leqslant n$, we have

$$L_n^{s^*}(g) \leqslant [L_n^s(g)]^{s^*/s}.$$

Therefore

$$h^3\,\mathrm{E}|X_k^*|^3 \exp(hrtB_n) \leqslant b_3\,\mathrm{E}[X_k^2 g^{s^*}(X_k)] / \sum_{k=1}^{n} \mathrm{E}[X_k^2 g^{s^*}(X_k)],$$

where b_3 is an appropriately chosen constant. Taking into account the inequalities given above, we get

$$\mathrm{E}[\exp(hX_k^*)] \leqslant \exp[h^2\sigma_k^2/2+b_2\sigma_k^2/B_n^2+b_3\mathrm{E}[X_k^2 g^{s^*}(X_k)]/\sum_{k=1}^{n}\mathrm{E}[X_k^2 g^{s^*}(X_k)],$$

and therefore

$$\prod_{k=1}^{n}\mathrm{E}[\exp(hX_k^*)] \leqslant \exp[(h^2B_n^2/2)+b_4].$$

Since $t \geqslant [2s^{-1}(1+s)\log_+(1/L_n^s(g))]^{1/2}$, we have

$$B_n^2 h^2/2 \leqslant \log_+(1/L_n^s(g))+2s(2+s)\log t+b_5.$$

Thus

$$\mathrm{E}[\exp(hS_n^*)] \leqslant b_6 t^{2s(2+s)}[L_n^s(g)]^{-1},$$

and the desired estimation follows from (2.1) and (2.2).

THEOREM 2. *There exist positive constants b and r (depending on s) such that for all $n \geqslant 1$ and every t with*

$$t^2 \leqslant 2s^{-1}(1+s)\log_+(1/L_n^s(g))$$

we have

$$\Delta_n(t) \leqslant b\exp[(\sigma-1)t^2/2]L_n^{s^*}(g)+T_n(r),$$

where $\sigma = s^*/2(1+s)$ and $s^* = \max\{s \geqslant z > 0\colon x/g^z(x)$ is nondecreasing$\}$.

Proof. We prove the theorem only for $t > 0$, as the proof for $t < 0$ is analogous. For $0 < t \leqslant 1$ the theorem follows immediately from Theorem 5 in [10], p. 112. Let $t > 1$ and let $d > 0$ be a real number to be determined later. Then

$$|P[S_n < tB_n]-\Phi(t)| \leqslant |P[S_n^* < tB_n]-\Phi(t)|+T_n(d),$$

where

$$S_n^* = \sum_{k=1}^{n} X_k I(|X_k| \leqslant dtB_n)$$

and

$$T_n(d) = \sum_{k=1}^{n} P[|X_k| > dtB_n].$$

Thus it is enough to show that

$$|P[S_n^* \leqslant tB_n]-\Phi(t)| \leqslant b\exp[(\sigma-1)t^2/2]L_n^{s^*}(g).$$

To this aim let us put $X_k^* = X_k I(|X_k| \leqslant dtB_n)$, $k = 1, 2, \ldots, n$, and for $1 \leqslant k \leqslant n$

$$h_k(t) = \mathrm{E}[\exp(tX_k^*/B_n)],$$

$$m_k(t) = h_k^{-1}(t)\mathrm{E}[X_k^*\exp(tX_k^*/B_n)],$$

$$M_n'(t) = \sum_{k=1}^{n} m_k(t), \qquad D_n^2(t) = \sum_{k=1}^{n} d_k^2(t),$$

where

$$[m_k^2(t)+d_k^2(t)]\,h_k(t) = \mathrm{E}\,[(X_k^*)^2\exp(tX_k^*/B_n)].$$

Furthermore, let

$$G_k(x) = \int_{-\infty}^{x} h_k^{-1}(t)\exp(tu/B_n)\,dP\,[X_k^* < u].$$

Then standard methods (see e.g. [4]) yield

$$P\,[S_n^* \geqslant tB_n] = A_n(t)\int_{[\ldots]} \exp(-ztM_n(t)B_n^{-1})\,dG_n^*(z),$$

where $[\ldots]$ denotes $\left[z > (tB_n - M_n(t))D_n^{-1}(t)\right]$ and

$$A_n(t) = \prod_{k=1}^{n} h_k(t)\exp(-tM_n(t)B_n^{-1}),$$

and

$$G_n^*(z) = G_1 * G_2 * \ldots * G_n\big(zD_n(t)+M_n(t)\big)$$

is the convolution of the distribution functions G_k, $1 \leqslant k \leqslant n$.

Note now that for every $k = 1, 2, \ldots, n$ the following inequalities are true:

$$\text{(2.3)} \qquad |\mathrm{E}\,X_k^*| \leqslant b_1\,\mathrm{E}\,[X_k^2 g^{s^*}(X_k)]/tB_n g^{s^*}(B_n),$$

$$\text{(2.4)} \qquad \exp(-dt^2) \leqslant h_k(t) \leqslant \exp(dt^2),$$

$$\text{(2.5)} \qquad 0 \leqslant \sigma_k^2 - \mathrm{E}\,(X_k^*)^2 \leqslant b_2\,\mathrm{E}\,[X_k^2 g^{s^*}(X_k)]/g^{s^*}(B_n),$$

$$\text{(2.6)} \qquad \mathrm{E}\,[|X_k^*|^3\exp(tX_k^*/B_n)] \leqslant b_3\exp(dt^2)\,\mathrm{E}\,[X_k^2 g^{s^*}(X_k)]\,tB_n/g^{s^*}(B_n).$$

Now, using (2.3)–(2.6), we have

$$\text{(2.7)} \qquad |h_k(t) - 1 - \sigma_k^2 t^2/2B_n^2| \leqslant b_4\exp(dt^2)\,\mathrm{E}\,X_k^2 g^{s^*}(X_k)/B_n^2 g^{s^*}(B_n),$$

$$\text{(2.8)} \qquad |h_k(t)\,m_k(t) - \sigma_k^2 t/B_n| \leqslant b_5\exp(dt^2)\,\mathrm{E}\,X_k^2 g^{s^*}(X_k)/B_n g^{s^*}(B_n),$$

$$\text{(2.9)} \qquad |\mathrm{E}\,(X_k^*)^2\exp(tX_k/B_n) - \sigma_k^2| \leqslant b_6\exp(dt^2)\,\mathrm{E}\,X_k^2 g^{s^*}(X_k)/g^{s^*}(B_n).$$

Furthermore, by the Berry–Essen inequality, Theorem 3 in [10] (p. 111), formulas (2.3), (2.4) and (2.6), we get

$$\text{(2.10)} \qquad \sup_{z\in\boldsymbol{R}}|G_n^*(z) - \Phi(z)| \leqslant b_7\exp(3dt^2)\,B_n g^{-s^*}(B_n)\,D_n^{-3}(t)\sum_{k=1}^{n}\mathrm{E}\,X_k^2 g^{s^*}(X_k).$$

But using (2.7)–(2.9) and (2.3)–(2.5), we have

$$\text{(2.11)} \qquad B_n^{-2}\,|D_n^2(t) - B_n^2| \leqslant b_8\exp(3dt^2)\,L_n^{s^*}(g).$$

Furthermore, taking into account (2.7)–(2.9) and the identity (38) in [11], p. 446, we can obtain the following inequalities:

$$\text{(2.12)} \qquad |A_n(t) - \exp(-t^2/2)| \leqslant b_9\exp(3dt^2)\,L_n^{s^*}(g),$$

$$\text{(2.13)} \qquad |M_n(t)\,B_n^{-1} - t| \leqslant b_{10}\exp(3dt^2)\,L_n^{s^*}(g).$$

On the other hand, by (2.10) and (2.13), we have

$$(2.14)\qquad \Big|\int_{[\ldots]} \exp(-tzM_n(t)B_n^{-1})\,d(G_n^*(z)-\Phi(z))\Big| \leqslant b_{11}\exp(3dt^2)\,L_n^{s^*}(g)$$

and

$$(2.15)\qquad \Big|\int_{[\ldots]} \exp(-tzM_n(t)B_n^{-1})\,d\Phi(z)-\exp(t^2/2)\,\Phi(-t)\Big| \leqslant b_{12}\exp(3dt^2)\,L_n^{s^*}(g),$$

where $[\ldots]$ denotes $\big[z > (tB_n - M_n(t))D_n^{-1}(t)\big]$. Thus, by (2.12), (2.14) and (2.15), we get

$$|P[S_n^* < tB_n] - \Phi(t)| \leqslant b\exp[(\sigma-1)t^2/2]\exp[(6d-\sigma)t^2/2]\,L_n^{s^*}(g).$$

The last inequality completes the proof of Theorem 2.

From Theorems 1 and 2 we can easily obtain a generalization of the Berry–Essen inequality obtained in Theorem 6 of [10], p. 115, and [3], [8], [9] and [6].

THEOREM 3. *There exists a constant b depending on s such that for all $n \geqslant 1$ and $t \in \boldsymbol{R}$*

$$\Delta_n(t) \leqslant bg^s(B_n)\big(L_n^s(g)+L_n^{s^*}(g)\big)/\big(1+t^2g^s(tB_n)\big).$$

Proof. For $|t| \leqslant 1$ the assertion follows from Theorem 5 of [10], p. 112. For $|t| > 1$ the proof of this theorem follows from Theorems 1 and 2.

From Theorems 1 and 2 we can also obtain the approximation of the moments of S_n/B_n of order greater than 2 by the corresponding moments of a standard normal distribution. This problem has been studied in [1], [2], [7], and [6].

THEOREM 4. *Suppose the function g is differentiable. Then there exists a constant $b > 0$ depending on s such that for all $n \geqslant 1$*

$$|\mathrm{E}\,[S_n^2 g^s(S_n)/B_n^2 g^s(B_n)] - \mathrm{E}\,[N^2 g^s(NB_n)/g^s(B_n)]| \leqslant b\big(L_n^s(g)+L_n^{s^*}(g)\big),$$

where N is a random variable with the standard normal distribution.

Proof. Since $S_n^2 g^s(S_n)/B_n^2 g^s(B_n) > 0$ implies

$$\mathrm{E}\,[S_n^2 g^s(S_n)/B_n^2 g^s(B_n)] = \Big(\int_0^\infty P[|S_n| > h^{-1}(t)B_n]\,dt\Big)/g^s(B_n),$$

where $h(t) = t^2 g^s(tB_n)$, $t \in \boldsymbol{R}$, and $s > 0$, we obtain

$$|\mathrm{E}\,[S_n^2 g^s(S_n)/B_n^2 g^s(B_n)] - \mathrm{E}\,[N^2 g^s(NB_n)/g^s(B_n)]| \leqslant \int_0^\infty \big|P[|S_n| > h^{-1}(t)B_n] - 2\Phi(h^{-1}(t))\big|\,dt/g^s(B_n).$$

Thus putting $a_n = 2s^{-1}(1+s)\log_+\left(1/L_n^s(g)\right)$, from Theorems 1 and 2 we get

$$\int_0^\infty \left|P[|S_n| > h^{-1}(t)B_n] - 2\Phi(h^{-1}(t))\right| dt$$

$$\leqslant b_1 L_n^{s^*}(g) \int_0^{a_n} \exp[-(1-\sigma)[h^{-1}(t)]^2/2]\, dt$$

$$+ b_2 L_n^s(g) \int_{a_n}^{\infty} [h^{-1}(t)]^{-2(2+s)}\, dt$$

$$+ \sum_{k=1}^{n} \int_0^\infty \{P[|X_k| \geqslant h^{-1}(t) r B_n]\}\, dt.$$

Since the integrals in the first two terms of this inequality are finite, the conclusion of Theorem 4 is proved, as

$$\sum_{k=1}^{n} \int_0^\infty \{P[|X_k| \geqslant h^{-1}(t) r B_n]\}\, dt \leqslant b L_n^s(g).$$

Now let us consider the speed of convergence to zero of $1 - F_n(t_n)$ under the assumption that $t_n \to \infty$ as $n \to \infty$.

THEOREM 5. *Let* $\{t_n,\ n \geqslant 1\}$ *be a sequence of real numbers such that* $t_n \to \infty$ *and*

$$t_n^2 \leqslant 2\log_+\left(1/L_n^s(g)\right) + \log_+\log_+\left(1/L_n^s(g)\right) + M, \tag{2.16}$$

where M *is a positive constant. If*

$$L_n^s(t_n, g) = \sum_{k=1}^{n} \mathrm{E}[X_k^2 g^s(X_k) I(|X_k| > t_n B_n)] / \sum_{k=1}^{n} \mathrm{E}[X_k^2 g^s(X_k)] \to 0 \tag{2.17}$$

as $n \to \infty$, *then*

$$1 - F_n(t_n) \sim (2\pi)^{-1/2} t_n^{-1} \exp(-t_n^2/2).$$

Proof. Without loss of generality we may and do assume that

$$t_n^2 \leqslant 2s^{-1}(1+s)\log_+\left(1/L_n^s(g)\right).$$

From Theorem 2 we get

$$t_n \exp(t_n^2/2)\,|P[S_n \geqslant t_n B_n] - \Phi(-t_n)| \leqslant b t_n \exp(t_n^2 \sigma/2) L_n^{s^*}(g)$$

$$+ g^s(B_n)/g^s(t_n B_n)\,[\exp(t_n^2/2) L_n^s(g) L_n^s(t_n, g)]\,(1/t_n).$$

Since $t_n^2 \leqslant 2s^{-1}(1+s)\log_+\left(1/L_n^s(g)\right)$, we have

$$t_n \exp(t_n^2 \sigma/2) L_n^s(g) = o(1).$$

It is easy to show that, for $t > 1$, $g(B_n)/g(tB_n) < 1$ and the function $t \to t^{-1}\exp(t^2/2)$ is nondecreasing. Then

$$\left(g(B_n)/t_n g(t_n B_n)\right) \exp(t_n^2/2) L_n^s(g) = O(1).$$

Thus, taking into account (2.17) and Lemma 2 of [5], p. 166, we obtain the proof of Theorem 5.

Let us observe that from Theorem 5 we can easily obtain the main results of Rubin and Sethuraman [12] on probabilities of moderate deviations under the much less restrictive moment conditions. Namely, if

$$B_n^2/n > 1 \quad \text{and} \quad \sup_n \Big((1/n) \sum_{k=1}^{n} \mathrm{E}\, X_k^2 g^s(X_k)\Big) < \infty,$$

then there exists a constant $M > 0$ such that

$$s \log_+ g^2(\sqrt{n}) \leqslant 2 \log_+ (1/L_n^s(g)) + M.$$

Therefore, if (2.17) holds, then from Theorem 5 with $t_n^2 = s \log_+ g^2(\sqrt{n})$ we get

$$P\big[|S_n| \geqslant B_n (s \log_+ g^2(\sqrt{n}))^{1/2}\big] \sim 2\big(g(\sqrt{n})\big)^{-s} \big(2\pi s \log_+ g^2(\sqrt{n})\big)^{-1/2}.$$

For sequences $\{t_n,\ n \geqslant 1\}$ with $t_n \to \infty$ which do not satisfy the assumptions of Theorem 5 we have the following

THEOREM 6. *Let* $\{t_n,\ n \geqslant 1\}$ *be a sequence of real numbers such that* $t_n \to \infty$ *and*

$$t_n^2 - 2 \log_+ (1/L_n^s(g)) - \log_+ \log_+ (1/L_n^s(g)) \to \infty \qquad \text{as } n \to \infty.$$

If (2.17) *holds, then*

$$(t_n^2 g^s(t_n B_n)/g^s(B_n))(1 - F_n(t_n)) = o(L_n^s(g)).$$

Proof. Let

$$t_n^2 = 2 \log_+ (1/L_n^s(g)) + \log_+ \log_+ (1/L_n^s(g)) + a_n,$$

where $a_n \to \infty$ as $n \to \infty$. From Theorem 1 for all $n \geqslant 1$ such that $t_n^2 \geqslant 2s^{-1}(1+s) \log_+ (1/L_n^s(g))$ we get

$$P[S_n \geqslant t_n B_n]$$
$$\leqslant \Phi(-t_n) + b_1 L_n^s(g)\, t_n^{-2(2+s)} + t_n^{-2} g^s(B_n)/g^s(t_n B_n)\, L_n^s(g)\, L_n^s(t_n, g),$$

where

$$\Phi(-t_n) \leqslant b_2 t_n^{-2(2+s)} L_n^s(g), \qquad g^s(t_n B_n)/g^s(B_n) \leqslant t_n^s.$$

On the other hand, by Theorem 2, for all $n \geqslant 1$ such that

$$t_n^2 \leqslant 2s^{-1}(1+s) \log_+ (1/L_n^s(g)),$$

we have

$$P[S_n \geqslant t_n B_n] \leqslant \Phi(-t_n) + b L_n^{s*}(g) \exp[(\sigma - 1) t_n^2/2]$$
$$+ t_n^{-2} g^s(B_n)/g^s(t_n B_n)\, L_n^s(g)\, L_n^s(t_n, g).$$

But, taking into account Lemma 2 of [5], p. 166, we get

$$(2.18)\qquad \big(t_n^2 g^s(t_n B_n)/g^s(B_n)\big)\Phi(-t_n) \leqslant bL_n^s(g)\exp(-a_n/2).$$

Furthermore

$$\big(t_n^2 g^s(t_n B_n)/g^s(B_n)\big) L_n^{s^*}(g)\exp[(\sigma-1)t_n^2/2]$$
$$\leqslant \big(bt_n g^s(t_n B_n)/g^s(B_n)\big)\exp(-t_n^2/2)\big(L_n^s(g)\big)^{s^*/2s}\big(\log_+(1/L_n^s(g))\big)^{1/2}.$$

Thus using (2.18) we obtain the desired assertion.

Let us observe that under the assumed moment conditions and the assumption (2.17) the property

$$t_n^2 - 2\log_+(1/L_n^s(g)) - \log_+\log_+(1/L_n^s(g)) \to \infty$$

characterizes the sequence $\{t_n,\ n \geqslant 1\}$ for which the assertion of Theorem 6 holds. The proof of this fact is an obvious modification of the proof of Remark 2 in [9].

Let us denote the L_p-norm $(1 \leqslant p \leqslant \infty)$ of any function f with respect to the Lebesgue measure by

$$\|f\|_p = \Big(\int_{\mathbf{R}} |f|^p d\mu\Big)^{1/p} \quad \text{and} \quad \|f\|_\infty = \sup_{x\in\mathbf{R}}|f(x)|.$$

THEOREM 7. *Assume the function g is differentiable. Then there exists a constant b (depending on s) such that for all $n \geqslant 1$ and every p $(1 \leqslant p \leqslant \infty)$*

$$(2.19)\qquad \big\|\Delta_n(t)\big[(h(t))^{p-1}h'(t)\big]^{1/p}\big\|_p \leqslant b\big(L_n^s(g)+L_n^{s^*}(g)\big),$$

where $h(t) = g^s(tB_n)t^2/g^s(B_n)$ and $h'(t)$ is the derivative of the function $h(t)$.

Proof. Let us observe that, by Theorem 3, (2.19) holds for $p=\infty$. On the other hand, if $1<p<\infty$, then

$$\big\|\Delta_n(t)\big[(h(t))^{p-1}h'(t)\big]^{1/p}\big\|_p \leqslant \|\Delta_n(t)h(t)\|_\infty^{\frac{p-1}{p}}\|\Delta_n(t)h'(t)\|_1^{1/p}.$$

Thus the proof will be completed if we show that

$$\|\Delta_n(t)h'(t)\|_1 \leqslant b\big(L_n^s(g)+L_n^{s^*}(g)\big).$$

Let $a_n(s) = 2s^{-1}(1+s)\log_+\big(1/L_n^s(g)\big)$. Then using Theorems 1 and 2 for $t \geqslant a_n(s)$ and $t \leqslant a_n(s)$, respectively, we get

$$\|\Delta_n(t)h'(t)\|_1 \leqslant b\big(L_n^s(g)+L_n^{s^*}(g)\big)+2\sum_{k=1}^{n}\int_0^\infty \{h'(t)P[|X_k|>rtB_n]\}\,dt.$$

Now we observe that

$$\sum_{k=1}^{n}\int_0^\infty \{h'(t)P[|X_k|>rtB_n]\}\,dt \leqslant bL_n^s(g).$$

The proof is completed.

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