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INDEPENDENCE WITH RESPECT TO A FAMILY OF MAPPINGS

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Introduction. In this note I consider a certain general notion of independence (announced in [5], p. 173) which contains, as special cases, some notions defined by Grätzer [1] (see also [2]), J. Schmidt [6], Świerczkowski [7] and myself [3] (see also [4] and [5]).

I adopt the definitions and notation of my quoted papers.

$\mathfrak{A} = (A; \boldsymbol{F})$ is a fixed (but arbitrary) algebra, letters a and b (with or without indices) denote always arbitrary elements of the carrier A and the letters f and g — arbitrary algebraic operations in $\mathfrak{A}$.

1. Extending of mapping to homomorphisms. Let us recall a proposition concerning these extensions:

(i) *Let p be a mapping of a non-void set $S \subset A$ into A. Then the following conditions are equivalent*:

(h) *there exists an extension of p to a homomorphism h of $C(S)$ into A,*

(a) *if $a_1, \ldots, a_{m+n} \in S$, $f \in A^{(m)}$, $g \in A^{(n)}$ $(m, n = 1, 2, \ldots)$ and*

$$f(a_1, \ldots, a_m) = g(a_{m+1}, \ldots, a_{m+n}),$$

then

$$f\big(p(a_1), \ldots, p(a_m)\big) = g\big(p(a_{m+1}), \ldots, p(a_{m+n})\big);$$

(a′) *if $a_1, \ldots, a_n \in S$, $f, g \in A^{(n)}$ $(n = 1, 2, \ldots)$ and*

$$(*) \qquad f(a_1, \ldots, a_n) = g(a_1, \ldots, a_n),$$

then

$$(**) \qquad f\big(p(a_1), \ldots, p(a_n)\big) = g\big(p(a_1), \ldots, p(a_n)\big);$$

(a″) *if $a_1, \ldots, a_n$ are distinct elements of S, $f, g \in A^{(n)}$ $(n = 1, 2, \ldots)$ and* (*), *then* (**).

For proofs see [4], p. 51-52.

2. Independence with respect to a family of mappings. Let us denote by $\boldsymbol{M}$ the family of all mappings p, whose domain ($\mathrm{dom}\, p$) and range ($\mathrm{ran}\, p$) are contained in A. In other words,

$$\boldsymbol{M} = \bigcup_{\emptyset \neq S \subset A} A^S.$$

Further, denote by $\boldsymbol{H}$ the family of all mappings $p \in \boldsymbol{M}$ extendable to a homomorphism of $C(\mathrm{dom}\, p)$ into A or, in other words, of all mappings $p \in \boldsymbol{M}$ satisfying (h).

For any subfamily $\boldsymbol{Q}$ of $\boldsymbol{M}$ a subset I of A is called *$\boldsymbol{Q}$-independent* (in symbols $I \in \boldsymbol{Ind}(A, \boldsymbol{Q})$ or, shortly, $I \in \boldsymbol{Ind}(\boldsymbol{Q})$), whenever every mapping of I belonging to $\boldsymbol{Q}$ can be extended to a homomorphism of $C(I)$ into A. Further, we say that a *set $S \subset A$ satisfies conditions* (A), (A′) *or* (A″) if any mapping p of S belonging to $\boldsymbol{Q}$ satisfies (a), (a′) or (a″), respectively. Proposition 1 (i) implies proposition

(i) *For any $S \subset A$ the conditions $S \in \boldsymbol{Ind}(\boldsymbol{Q})$,* (A), (A′) *and* (A″) *are equivalent.*

The following propositions are easily deduced from definition of $\boldsymbol{Ind}(\boldsymbol{Q})$:

(ii) *If $\boldsymbol{Q}_1 \subset \boldsymbol{Q}_2 \subset \boldsymbol{M}$, then $\boldsymbol{Ind}(\boldsymbol{Q}_2) \subset \boldsymbol{Ind}(\boldsymbol{Q}_1)$.*

(iii) *For any $\boldsymbol{Q} \subset \boldsymbol{M}$ we have*

$$\boldsymbol{Ind} = \boldsymbol{Ind}(\boldsymbol{M}) \subset \boldsymbol{Ind}(\boldsymbol{Q}) \subset \boldsymbol{Ind}(\boldsymbol{H}) = 2^A.$$

(iv) *Let $\boldsymbol{Q} \subset \boldsymbol{M}$ and suppose that for every $S \subset T \subset A$ and every $p \in \boldsymbol{Q} \cap A^S$ there is a $q \in \boldsymbol{Q} \cap A^T$ such that $q|S = p$. Then the family $\boldsymbol{Ind}(\boldsymbol{Q})$ is hereditary.*

Now I shall prove proposition

(v) *Let $\boldsymbol{Q} \subset \boldsymbol{M}$ and suppose that for every $S \subset T \subset A$ and every $q \in \boldsymbol{Q} \cap A^T$ we have $q|S \in \boldsymbol{Q}$. Then, if every finite subset of a set I is $\boldsymbol{Q}$-independent, then I is $\boldsymbol{Q}$-independent.*

Let $p \in \boldsymbol{Q} \cap A^I$. In view of 2(i) it suffices to prove that p satisfies (a′). We suppose

$$S = \{a_1, \ldots, a_n\} \subset I$$

and

$$f(a_1, \ldots, a_n) = g(a_1, \ldots, a_n).$$

Hence, by hypothesis, $p|S \in \boldsymbol{Q}$ and, consequently, by hypothesis on finite subsets of I, we have

$$f\big(p(a_1), \ldots, p(a_n)\big) = g\big(p(a_1), \ldots, p(a_n)\big), \qquad \text{q.e.d.}$$

3. $\boldsymbol{S}$-independence and $\boldsymbol{S}_0$-independence. Let us denote by $\boldsymbol{S}$ the family of all mappings $p \in \boldsymbol{M}$ with $\mathrm{ran}\, p \subset C(\mathrm{dom}\, p)$ and by $\boldsymbol{S}_0$ the family of all mappings $p \in \boldsymbol{M}$ with $\mathrm{ran}\, p \subset \mathrm{dom}\, p$.

The S-independence (in the sense of section 2) coincides with the independence "in sich", considered by Schmidt [6] and the S_0-independence — with the weak independence considered by Świerczkowski [7]. In view of the proposition of the preceding section we have

(i) $\mathbf{Ind} \subset \mathbf{Ind}(S) \subset \mathbf{Ind}(S_0)$.

(ii) *Families* $\mathbf{Ind}(S)$ *and* $\mathbf{Ind}(S_0)$ *are hereditary.*

Moreover it is easy to check that

(iii) *If* $\mathfrak{A}$ *is functionally complete, or, more generally, if every element of* A *is an algebraic constant in* $\mathfrak{A}$, *then* $\mathbf{Ind}(S) = \mathbf{Ind} = \{\emptyset\}$.

In fact, in view of (ii) it suffices to prove that no one-element set $\{c\}$ belongs to $\mathbf{Ind}(S)$. Since, by hypothesis, $C(\{c\}) = A$, every mapping $c \to a \in A$ belongs to S. So, for one-element sets, the independence and the S-independence are equivalent and, as we know, no algebraic constant is independent.

Proposition (iii) is false for S_0-independence: see below 5 (ii).

4. A certain quasi-order and G-independence (1). We write $a \succ b$, whenever for any $f, g \in A^{(1)}$ if $f(a) = g(a)$, then $f(b) = g(b)$. More generally, $(a_1, \ldots, a_n) \succ (b_1, \ldots, b_n)$ whenever for any $f, g \in A^{(n)}$ if $f(a_1, \ldots, a_n) = g(a_1, \ldots, a_n)$, then $f(b_1, \ldots, b_n) = g(b_1, \ldots, b_n)$. The so defined relation is of course reflexive. Moreover, we obviously have

(i) *If* $c \succ a$, *where* c *is an algebraic constant, then* $a = c$.

(ii) *If* A *is a group with the unity* e, *then* 1° $a \succ e$ *for every* a, *and* 2° *for any* a *and* b *of finite order we have* $a \succ b$ *iff the order of* a *is divisible by the order of* b.

A mapping $p: S \to A$, where $S \subset A$, is *diminishing* if $a \succ p(a)$ for every $a \in S$. It is *totally diminishing* if $(a_1, \ldots, a_n) \succ \big(p(a_1), \ldots, p(a_n)\big)$ for every finite sequence $a_1, \ldots, a_n \in S$. Evidently this is another form of condition (a′) of section 1, whence

(iii) *A mapping* p *of* $S \subset A$ *into* A *can be extended to a homomorphism of* $C(S)$ *into* A *iff* p *is totally diminishing.*

The class of all diminishing mappings will be denoted by G. The following proposition concerning the G-independence (in the sense of section 2) is an easy consequence of 2 (i) and 4 (iii):

(iv) *Let* I *be a subset of* A. *Then the following conditions are equivalent:*

(g) I *is* G*-independent*;

(1) This notion has been introduced (under the name of *weak independence*) by Grätzer (see [1] and [2]). Here I consider only individual algebras and I write $a \succ b$, while Grätzer considers equational classes and the relation $0(a) \leqslant 0(b)$. All propositions of this section are (explicitly or implicitly) contained in [1].

(g′) *every diminishing mapping of I into A is totally diminishing*;

(g″) *for any different elements $a_1, \ldots, a_n$ of I, for any elements $b_1, \ldots, b_n$ of I such that $a_j \succ b_j$ (for $1, \ldots, n$), and for every $f, g \in A^{(n)}$, if*

$$f(a_1, \ldots, a_n) = g(a_1, \ldots, a_n),$$

then

$$f(b_1, \ldots, b_n) = g(b_1, \ldots, b_n).$$

Proposition (iv) implies proposition

(v) *The family $\mathbf{Ind}(G)$ is hereditary and any set $I \subset A$ is G-independent iff every finite subset of I is G-independent.*

Proposition (v) may also be deduced from propositions 2 (iv) and 2 (v).

We shall now prove proposition (see [1], p. 232)

(vi) *If $\mathfrak{A} = (A; x+y, -x)$ is an Abelian group, then a set $I \setminus \{0\} \subset A$ is G-independent iff I is linearly independent, i.e. if for every $a_1, \ldots, a_n \in I$ the equation*

$$\sum_{j=1}^{n} k_j a_j = 0$$

implies $k_j a_j = 0$ for $j = 1, 2, \ldots, n$.

Let us suppose that $I \setminus \{0\}$ is linearly independent and, moreover, that

$$a_1, \ldots, a_n, b_1, \ldots, b_n \in I, \quad a_j \succ b_j \text{ for } j = 1, \ldots, n, \tag{1}$$

and

$$f(a_1, \ldots, a_n) = g(a_1, \ldots, a_n). \tag{2}$$

Operations f and g, as algebraic in $\mathfrak{A}$ are of the form

$$f(x_1, \ldots, x_n) = \sum_j k_j x_j, \quad g(x_1, \ldots, x_n) = \sum_j l_j x_j.$$

It follows from (2) that

$$\sum_j (k_j - l_j) a_j = 0,$$

whence, by the linear independence of $I \setminus \{0\}$, we obtain $(k_j - l_j) a_j = 0$ then, by (1), $(k_j - l_j) b_j = 0$, and, consequently,

$$f(b_1, \ldots, b_n) = g(b_1, \ldots, b_n).$$

Thus the set I satisfies (g″), and by (iv), I is G-independent.

Let us suppose now that I is G-independent, and, moreover, that

$$a_1, \ldots, a_n \in I \quad \text{and} \quad \sum_j k_j a_j = 0.$$

Putting

$$f(x_1, \ldots, x_n) = \sum_j k_j x_j \quad \text{and} \quad g(x_1, \ldots, x_n) = 0$$

we get

(3) $$f(a_1, \ldots, a_n) = g(a_1, \ldots, a_n).$$

Let us fix j_0 such that $1 \leqslant j_0 \leqslant n$, and put $b_{j_0} = a_{j_0}$ and $b_j = 0$ for $j \neq j_0$. Then $a_j \succ b_j$ for $j = 1, 2, \ldots, n$ and, on account of (3), of the G-independence of I and of proposition (iv) (condition (g″)), we obtain

$$k_{j_0} a_{j_0} = \sum_j k_j b_j = f(b_1, \ldots, b_n) = g(b_1, \ldots, b_n) = 0.$$

5. Independence of one-point sets. As we know, for some families $\boldsymbol{Q}$ all one-point subsets of A are $\boldsymbol{Q}$-independent. We can now give an obvious necessary and sufficient condition:

(i) *$\{a\} \in \boldsymbol{Ind}(\boldsymbol{Q})$ iff every mapping $p: \{a\} \to A$ belonging to $\boldsymbol{Q}$ is diminishing.*

Of course, this condition is satisfied for every a by the family $\boldsymbol{G}$. It is also satisfied by the family $\boldsymbol{S}_0$, since, if the mapping $p: \{a\} \to A$ belongs to $\boldsymbol{S}_0$, we have, by definition of $\boldsymbol{S}_0$, $p(a) = a$, whence p is diminishing. Hence

(ii) *Every one-point set is $\boldsymbol{G}$-independent* ([1], p. 232) *and $\boldsymbol{S}_0$-independent.*

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