COLLOQUIUM MATHEMATICUM

VOL. LIX 1990 FASC. 1

COUNTABLE PRODUCTS OF LCA GROUPS: THEIR CLOSED SUBGROUPS, QUOTIENTS AND DUALITY PROPERTIES

BY

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The Pontryagin–van Kampen duality theorem asserts that if G is a locally compact abelian group (shortly, an LCA group), then the dual group $G^\wedge$ is also locally compact and the canonical embedding $G \to G^{\wedge\wedge}$ is a topological isomorphism. Hence it follows easily that the duality between G and $G^\wedge$ induces dualities between appropriate closed subgroups and Hausdorff quotient groups of G and $G^\wedge$ – we say that the duality between G and $G^\wedge$ is strong.

The possibility of extending the Pontryagin–van Kampen theorem to certain abelian topological groups which are not locally compact was investigated in several papers. A brief survey can be found in [4], in the Notes after Sections 23–25; see also [1], [2], [5]–[9], [11], [12] and [14]–[16]. In the present paper we are concerned with the strong duality between countable products and direct sums of LCA groups.

To formulate the result we have to introduce several notions. Let G be an abelian topological group and $G^\wedge$ its character group endowed with the compact-open topology. Here by a *character* we mean a homomorphism of G into the additive group $\boldsymbol{T} := \boldsymbol{R}/\boldsymbol{Z}$. We say that G is a *reflexive group* if the canonical evaluation map is a topological isomorphism of G onto $G^{\wedge\wedge}$.

Let H be a subgroup of G. We say that H is *dually embedded* in G if each continuous character of H can be extended to a continuous character of G. The set

$$\{\chi \in G^\wedge : \chi(g) = 0 \text{ for all } g \in H\}$$

is a closed subgroup of $G^\wedge$; we denote it by H^0. We say that H is *dually closed* in G if, for each $g \in G \setminus H$, there exists some $\chi \in H^0$ with $\chi(g) \neq 0$.

A reflexive group G is said to be *strongly reflexive* if the duality between G and $G^{\wedge\wedge}$ induces dualities between appropriate closed subgroups and Hausdorff quotient groups of G and $G^\wedge$ (in the terminology of [2], if the canonical pairing $G \times G^\wedge \to \boldsymbol{T}$ is a strong duality). This is equivalent to the assertion that each closed subgroup of G and of $G^\wedge$ is dually closed and dually embedded, and that each closed subgroup and Hausdorff quotient group of G and of $G^\wedge$ is reflexive (cf. [2], Proposition 12). If H is a closed subgroup of a strongly reflexive group G, then the canonical mappings

$$G^\wedge/H^0 \to H^\wedge \quad \text{and} \quad (G/H)^\wedge \to H^0$$

are topological isomorphisms. Moreover, H and G/H are both strongly reflexive ([2], Proposition 13). Many examples of reflexive groups which are not strongly reflexive are known; such are, for instance, all infinite dimensional Banach spaces treated as additive topological groups (cf. [1], Remark 2.10).

Let $(A_n)_{n=1}^\infty$ be a sequence of abelian topological groups. By $\prod_{n=1}^\infty A_n$ we shall denote their product endowed with the usual product topology. By $\sum_{n=1}^\infty A_n$ we shall denote their direct sum endowed with the so-called rectangular topology. Algebraically, $\sum_{n=1}^\infty A_n$ is a subgroup of $\prod_{n=1}^\infty A_n$ consisting of sequences with almost all coordinates equal to zero. As a basis of neighbourhoods of zero in $\sum_{n=1}^\infty A_n$ one may take the family of sets of the form

$$\sum_{n=1}^\infty A_n \cap \prod_{n=1}^\infty U_n,$$

where U_n is a neighbourhood of zero in A_n for every n (cf. [2], p. 20). The definition of the direct sum of uncountably many abelian topological groups is somewhat more complicated (see [6], p. 652).

Now, let $(A_n)_{n=1}^\infty$ be a sequence of LCA groups. Let us put

$$G = \prod_{n=1}^\infty A_n \quad \text{and} \quad H = \sum_{n=1}^\infty A_n.$$

Let P and Q be arbitrary closed subgroups of G and H, respectively.

Kaplan [7] proved that P is dually closed and dually embedded in G. He also proved that limits of direct and inverse sequences of LCA groups are reflexive (for the definition of the limit of a direct system of abelian topological groups, we refer the reader to [7], p. 421). Varopoulos [15] showed that Q is dually closed and dually embedded in H, and that the mappings

$$(H/Q)^\wedge \to Q^0 \quad \text{and} \quad H^\wedge/Q^0 \to Q^\wedge$$

are both topological isomorphisms. Noble [12] proved that P is reflexive. His results also imply the continuity of the canonical mappings

$$G/P \to (G/P)^{\wedge\wedge}, \quad G \to G^{\wedge\wedge} \quad \text{and} \quad H/Q \to (H/Q)^{\wedge\wedge}.$$

Brown et al. [2] proved that G is strongly reflexive if A_1 is compact and A_n

is isomorphic to $\boldsymbol{R}$ or $\boldsymbol{Z}$ for $n \geqslant 2$ (or, dually, if A_1 is discrete and A_n is isomorphic to $\boldsymbol{R}$ or $\boldsymbol{T}$ for $n \geqslant 2$). Venkataraman [16] showed that P is reflexive and that the mapping $G^\wedge/P^0 \to P^\wedge$ is a topological isomorphism.

The above results admit of the following common generalization:

THEOREM. *Let $(A_n)_{n=1}^{\infty}$ be a sequence of LCA groups. Then the groups $\prod_{n=1}^{\infty} A_n$ and $\sum_{n=1}^{\infty} A_n$ are both strongly reflexive, the dual groups being topologically isomorphic to $\sum_{n=1}^{\infty} A_n^\wedge$ and $\prod_{n=1}^{\infty} A_n^\wedge$, respectively.*

As we shall see below, this theorem is an almost immediate consequence of the results of [7] and [15]. Most likely, this fact has nowhere been explicitly stated.

Proof. Without the word "strongly", the result follows from Kaplan's duality theorem (see [6], p. 655). Put

$$G = \prod_{n=1}^{\infty} A_n, \quad H = \sum_{n=1}^{\infty} A_n$$

and let P and Q be arbitrary closed subgroups of G and H, respectively. We have to show that P and Q are dually closed and dually embedded in G and H, respectively, and that all the four groups P, Q, G/P and H/Q are reflexive.

That P is dually embedded and dually closed in G follows from Theorems 1 and 2 of [7]. That Q is dually embedded and dually closed in H is a consequence of [15], the theorem on p. 509, part (B), and a refinement of (B) given at the top of p. 512. Kaplan proved that limits of direct and inverse sequences of LCA groups are reflexive (see [7], the theorem on p. 425). Thus, to complete the proof, we only have to show that P and G/P (resp., Q and H/Q) can be represented as limits of inverse (resp., direct) sequences of LCA groups.

Let us put

$$G_n = A_1 \times \ldots \times A_n \quad (n = 1, 2, \ldots);$$

let π_n: $G \to G_n$ be the canonical projection and let P_n be the closure of $\pi_n(P)$ in G_n. If $m \geqslant n$, then the canonical projection

$$\pi_{mn} : G_m \to G_n$$

induces the canonical projections

$$\pi'_{mn} : P_m \to P_n \quad \text{and} \quad \pi''_{mn} : G_m/P_m \to G_n/P_n.$$

The limit of the inverse system $\{\pi_{mn} : G_m \to G_n\}$ may be identified with G. It is also clear that the limit of the inverse system $\{\pi'_{mn} : P_m \to P_n\}$ may be identified with P (cf. [3], Corollary 2.5.7). Similarly, the limit of the inverse system

$$\{\pi''_{mn} :\ G_m/P_m \to G_n/P_n\}$$

is topologically isomorphic to G/P (a sketch of the proof can be found in [15], at the bottom of p. 513).

Let us now put

$$H_n = A_1 \oplus \ldots \oplus A_n \quad (n = 1, 2, \ldots).$$

We may identify H_n with a subgroup of H; let $Q_n = Q \cap H_n$. If $m \leqslant n$, then the identity embedding

$$\varrho_{mn}: H_m \to H_n$$

induces the canonical monomorphisms

$$\varrho'_{mn}: Q_m \to Q_n \quad \text{and} \quad \varrho''_{mn}: H_m/Q_m \to H_n/Q_n.$$

We may identify the limit of the direct system $\{\varrho_{mn}: H_m \to H_n\}$ with H. Next, we may identify the limit of the direct system $\{\varrho'_{mn}: Q_m \to Q_n\}$ with Q (see [15], Proposition 3(A), p. 476). Finally, the limit of the direct system

$$\{\varrho''_{mn}: H_m/Q_m \to H_n/Q_n\}$$

is easily seen to be topologically isomorphic to H/Q (cf. [15], the remarks at the top of p. 512).

The above proof is not elementary. At any rate, the proof that Q is dually closed and dually embedded in H, given in [15], requires the application of group algebras. However, one can obtain a relatively elementary proof of the Theorem, following the way of [2]. Such a proof would have to be based on the structure theorem for LCA groups, but relatively elementary proofs of the latter can be found in [10] and [13].

Leptin [9] gave an example of a non-reflexive closed subgroup of an uncountable product of LCA groups. On the other hand, Negrepontis proved in [11] that the limit of an arbitrary direct or inverse system of compactly generated LCA groups is reflexive. The proof given there includes, however, an error and it is not known if the whole theorem is true at all; the error lies in the proof of Proposition 4.4 on p. 250, saying that a closed subgroup of a direct sum of LCA groups is dually closed. The author does not know if a closed subgroup of an uncountable direct sum of real lines must be dually closed or dually embedded. **(P 1380)**

Applying a completely different and much more complicated method, the author proved in [1] that nuclear Fréchet spaces, treated as additive topological groups, are strongly reflexive. Hence and from the structure theorem for LCA groups it is not difficult to infer that the product of a nuclear Fréchet space and countably many LCA groups is strongly reflexive.

Acknowledgement. The author wishes to thank the referee for the remarks which contributed to a considerable improvement of the text of the paper.

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Reçu par la Rédaction le 8.1.1988;
en version modifiée le 20.7.1988