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A SOLUTION OF A PROBLEM OF B. ROTMAN

BY

E. GRZEGOREK (WROCŁAW)

The following proposition gives the negative answer to the question of Rotman ([1]) (P 937).

PROPOSITION. *Let M be an infinite set of power a. Then there exists a family $\{f_i: 0 \leqslant i < a\}$ of bijections of M onto M with the following properties:*

(a) *for all $i < j < a$ we have*

$$F_i = \emptyset \quad \text{and} \quad |M \setminus F_{ij}| = a,$$

where

$$F_i = \{x \in M : f_i(x) = x\} \quad \text{and} \quad F_{ij} = \{x \in M : f_i(x) = f_j(x)\};$$

(b) *there is no subset $X \subseteq M$ of power a such that*

$$|f_0(X) \cap f_1(X)| < a \quad \text{and} \quad |f_0(X) \cap f_2(X)| < a.$$

Proof. Let $\{M_k: k < 8\}$ be a partition of M such that $|M_k| = a$ for all $k < 8$. Let f_0, f_1, and f_2 be bijections of M such that

$$f_1(x) = f_0(x) \quad \text{for every } x \in \bigcup \{M_k: k < 4\},$$
$$f_2(x) = f_0(x) \quad \text{for every } x \in \bigcup \{M_k: k > 3\},$$
$$f_0(M_0 \cup M_1) = M_2 \cup M_3, \quad f_0(M_2 \cup M_3) = M_0 \cup M_1,$$
$$f_0(M_4 \cup M_5) = M_6 \cup M_7, \quad f_0(M_6 \cup M_7) = M_4 \cup M_5,$$
$$f_1(M_4) = M_5, \quad f_1(M_5) = M_4, \quad f_1(M_6) = M_7, \quad f_1(M_7) = M_6,$$
$$f_2(M_0) = M_1, \quad f_2(M_1) = M_0, \quad f_2(M_2) = M_3, \quad f_2(M_3) = M_2.$$

Assume that M is ordered in the type a. We define, by transfinite induction on i, a family $\{f_i : i < a\}$ of bijections of M such that f_0, f_1, and f_2 are as at the beginning of the proof, and for every $2 < i < a$ and every $x \in M$ we have $f_i(x) \in M_{ix}$, where

$$M_{ix} = M \setminus \big(\{x\} \cup \{f_i(x'): x' < x\} \cup \{f_{i'}(x): i' < i\}\big).$$

([1]) See B. Rotman, *Correction to the paper "A theorem on almost disjoint sets"*, Colloquium Mathematicum 32 (1975), p. 307-308.

It is easy to check that the family $\{f_i: i < a\}$ with the above properties satisfies conditions (a) and (b) of the Proposition. Now, all we need to complete the proof is to define such a family of bijections. Let $2 < i < a$ and assume that the bijections $f_{i'}$ for $i' < i$ have been defined. In order to define a bijection f_i we determine, by transfinite induction with respect to order in M, the value $f_i(x)$ for $x \in M$. Let $x \in M$ and assume that $f_i(x')$ have been defined for $x' < x$. Let $f_i(x)$ be the smallest element of M_{ix}. It remains to prove that f_i is a bijection. It is clear that f_i is an injection. Thus it is enough to prove that f_i is a surjection. Suppose, on the contrary, that f_i is not a surjection, i.e., that there exists $x_0 \in M$ such that $f_i(x) \neq x_0$ for every $x \in M$. Put

$$A = \{y: \exists(i' < i)(f_{i'}(y) = x_0)\} \cup \{x_0\}.$$

It is easy to see that $x_0 \in M_{ix}$ for every $x \in M \setminus A$. Hence, since $f_i(x) = \min M_{ix}$, we have $f_i(x) \leqslant x_0$ for every $x \in M \setminus A$. Thus $|f_i(M \setminus A)| < a$. On the other hand, since f_i is an injection, we have

$$|f_i(M \setminus A)| = |M \setminus A|.$$

We have $|A| < a$, since $f_{i'}$ is an injection for every $i' < i$. Hence $|M \setminus A| = a$. Therefore $|f_i(M \setminus A)| = a$. Since we have shown that $|f_i(M \setminus A)| < a$, we have a contradiction.

Remark. It is also possible to prove the Proposition without transfinite induction.

INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES, WROCŁAW

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