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TAUBERIAN THEOREMS FOR CESÀRO SUMS (ADDENDUM AND CORRIGENDUM)

BY

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This note is concerned with certain additions and corrections to the author's paper [3] (¹) with the above title, for which he is indebted to Professor C. T. Rajagopal and Dr. B. Kuttner. The correction to the proof of Corollary I_2 (²), involving the additions in Section 1, are due to the former, while the comments and corrections of Sections 2 and 3 are due to the latter.

1. The deduction of Corollary I_2 from Theorem I is invalid since it makes the choice $W(x) \equiv 1$ in contradiction to the unboundedness of $W(x)$ assumed in hypothesis (3) (i) of Theorem I. However, Corollary I_2 may be deduced from Theorem I′ of this note, which is complementary to Theorem I and is proved by means of

LEMMA A′. *Let us define*

$$d_n = \frac{\Gamma(n+r)}{\Gamma(n+1)}, \quad 0 < r < 1,$$

and let $\{s_n\}$ satisfy the condition

$$S_n^1 \equiv s_0+s_1+\ldots+s_n \to 0 \quad as\ n \to \infty.$$

Let $h = h(n) < n$ be a positive integer such that h and $n-h$ both tend to ∞ with n. Then

$$\sum_{\nu=0}^{n-h} d_{n-\nu} s_\nu = o(h^{r-1}) \quad as\ n \to \infty.$$

(¹) Numbers in square brackets relate to the references at the end of this note. The author takes this opportunity to mention that the contents of [3], with the rectification of the proof of Corollary I_2 detailed in Section 1 of this note, supplied the material for Chapter II of a thesis entitled "Studies in summation processes" approved for the Ph. D. degree of the University of Madras.

(²) Throughout this note, the notation of [3] is preserved, and references to theorems, corollaries and steps of [3] are by their numbers in [3].

Proof. The proof is suggested by Hardy ([2], p. 101, proof of Theorem 45).

Writing $S_n^1 = S_n$ for convenience, we get by partial summation

$$\sum_{\nu=0}^{n-h} d_{n-\nu} s_\nu = S_0(d_n - d_{n-1}) + \ldots + S_{n_0}(d_{n-n_0} - d_{n-n_0-1}) + \ldots + \\ + S_{n-h-1}(d_{h+1} - d_h) + S_{n-h} d_h.$$

Since d_n is monotonic decreasing, we get further

$$\left|\sum_{\nu=0}^{n-h} d_{n-\nu} s_\nu\right| \leqslant |S_0|(d_{n-1} - d_n) + \ldots + |S_{n_0}|(d_{n-n_0-1} - d_{n-n_0}) + \ldots + \\ + |S_{n-h-1}|(d_h - d_{h+1}) + |S_{n-h}| d_h \\ < \max_{0 \leqslant \nu \leqslant n_0-1} |S_\nu|(d_{n-n_0} - d_n) + 2 \max_{n_0 \leqslant \nu \leqslant n-h} |S_\nu| d_h.$$

Since $S_n \to 0$ and $d_n \sim n^{r-1}$ as $n \to \infty$, given any small $\varepsilon > 0$, we can choose n_0 and h_0 so that

$$|S_n| < \varepsilon \text{ for } n \geqslant n_0, \quad d_h < (1+\varepsilon) h^{r-1} \text{ for } h \geqslant h_0.$$

Also, since $n-h$ tends to ∞ with n, we may suppose that $n-h > n_0$ and obtain, for all sufficiently large n (and h),

$$\left|\sum_{\nu=0}^{n-h} d_{n-\nu} s_\nu\right| < \text{const}\cdot(d_{n-n_0} - d_n) + 2\varepsilon(1+\varepsilon) h^{r-1}.$$

Now $d_n - d_{n-1} = O(n^{r-2})$ as $n \to \infty$, and so

$$d_{n-n_0} - d_n = (d_{n-n_0} - d_{n-n_0+1}) + \ldots + (d_{n-1} - d_n) \\ < \text{const}\cdot[(n-n_0)^{r-2} + \ldots + (n-1)^{r-2}] < \text{const}\cdot n^{r-2}.$$

Hence

$$\left|\sum_{\nu=0}^{n-h} d_{n-\nu} s_\nu\right| < \text{const}\cdot n^{r-2} + 2\varepsilon(1+\varepsilon) h^{r-1} < \text{const}\cdot h^{r-2} + 2\varepsilon(1+\varepsilon) h^{r-1}.$$

Since $h^{r-2} = o(h^{r-1})$ when n (as also h) tends to ∞, the proof of Lemma A′ is complete.

The following theorem may now be proved.

THEOREM I′. *If, in the notation of* [3],

$$S_n^1 = o(1), \quad s_n = O\{V(n)\},$$

where $V(x)$ is a positive monotonic decreasing function of $x > 0$, which tends to 0 as x tends to ∞, then, for any r such that $0 < r < 1$, we have

$$S_n^r = o[\{V(n)\}^{1-r}] \quad \textit{as } n \to \infty.$$

Proof. The proof of Theorem I with $\delta = 1$ is applicable except for (8) where now we have only

$$\max_{0 \leqslant \mu \leqslant n} |S_\mu^1| = O(1)$$

with O not replaceable by o (as we require). However, we may now use Lemma A′, taking h = integer next above $\varepsilon / V(n)$ (which integer tends to ∞ with n since $V(n)$ tends to 0). The resulting estimate is that, for all large n,

$$|T_1| \equiv \Big|\sum_{\nu=h}^{n} d_\nu s_{n-\nu}\Big| < \varepsilon h^{r-1} < \varepsilon\{\varepsilon/V(n)\}^{r-1} = \varepsilon^r\{V(n)\}^{1-r}.$$

On the other hand, our estimate of T_2 in step (9) of the proof of Theorem I now gives us, for all large n,

$$|T_2| \equiv \Big|\sum_{\nu=0}^{h-1} d_\nu s_{n-\nu}\Big| < K' h^r V(n) < K' \varepsilon^r \{V(n)\}^{1-r}.$$

From the above two steps we get the required result that, for all large n,

$$|S_n^r| \leqslant |T_1| + |T_2| < \mathrm{const} \cdot \varepsilon^r\{V(n)\}^{1-r}.$$

Theorem I′ with $V(x) \equiv 1/x$, applied to $s_n^* = s_n - s_{n-1}$ gives Corollary I_2 of [3] due to Hardy and Littlewood.

2. Corresponding to the hypothesis that $V(n) \to 0$ as $n \to \infty$ in Theorem I′ above, it was tacitly assumed in both Theorems I and II of [3] that

$$\frac{W(n)}{V(n)} \to \infty \quad \text{as } n \to \infty,$$

this assumption being required in the choice of h in the proof of Theorem I and in the choice of k in the proof of Theorem II.

In Theorems I and II of [3], the case

$$\frac{W(n)}{V(n)} = O(1) \quad \text{as } n \to \infty$$

is trivial and may be ignored, since then the conclusion (6) of Theorem I follows from (4) *alone* and the conclusion (13) of Theorem II follows from (11) *alone*. To dispose of the case $W(n)/V(n) = O(1)$, we note that in extension of (1) of [3],

$$S_n^{\varrho+s} = \sum_{\nu=0}^{n} A_{n-\nu}^{s-1} S_\nu^\varrho, \qquad \varrho, s, \varrho+s \geqslant -1,$$

and make choices of ϱ and s, relevant to Theorems I and II. The subsequent procedure is omitted, being similar to that in the corresponding case of Theorem I of [1].

Hypothesis (3) (ii) of Theorem I of [3] may be weakened to

$$\text{(3) (ii}') \qquad \frac{V(x')}{V(x)} < H \quad \text{if } 0 < x - x' < \eta x (\eta < 1)$$

which is the actual assumption involved in step (9) of the proof of Theorem I. With this weakening of hypothesis (3) (i) of Theorem I, there is no need to distinguish the cases (a) and (b) in Corollary I_3, case (b) being included in hypothesis (3) (ii′). A similar weakening of the corresponding hypothesis (10) (iii) of Theorem II of [3] cannot, however, be effected.

3. The following are corrections needed in [3].

In the proof of Theorem II (page 106, line 3) $d_\nu = \nu/(\nu-1+\delta)$ is monotonic decreasing in ν, the statement $d_\nu = (\nu-1+\delta)/\nu$ is monotonic increasing being a mistake. Consequently, the steps in page 106, lines 5 and 6 should be altered as follows:

$$\left|\sum_{\nu=1}^{k} \frac{\Gamma(\nu-1+\delta)}{\Gamma(\nu)} S_{N-\nu}^{p+1}\right| \leqslant 2\Gamma(\delta)\frac{1}{\delta}\max_{0\leqslant\mu\leqslant N}|S_\mu^{p+1+\delta}|$$
$$= o\{W(n+1)\} \quad \text{as } n \to \infty.$$

In page 106, lines 9 and 2 from the bottom, "innermost sum" should of course read "innermost summand".

O and O_L in line 10 of page 107 are to be corrected to o and o_L, respectively.

REFERENCES

[1] A. L. Dixon and W. L. Ferrar, *On Cesàro sums*, Journal of the London Mathematical Society 7 (1932), p. 87-93.

[2] G. H. Hardy, *Divergent series*, Oxford 1949.

[3] M. S. Rangachari, *Tauberian theorems for Cesàro sums*, Colloquium Mathematicum 11 (1963), p. 101-108.

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