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SINGULAR SETS OF SEPARATELY ANALYTIC FUNCTIONS

BY

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0. Introduction. Let Ω be an open set in $\mathbf{R}^{n_1} \times \ldots \times \mathbf{R}^{n_s}$, where $n_1, \ldots, n_s$ are positive integers. We say that a function

$$f : \Omega \ni (x^1, \ldots, x^s) \to f(x^1, \ldots, x^s) \in \mathbf{C}$$

is *p-separately analytic* (where $1 \leq p < s$) if for every point $(x_0^1, \ldots, x_0^s)$ in Ω and for every sequence $1 \leq i_1 < \ldots < i_p \leq s$ the function

$$(x^{i_1}, \ldots, x^{i_p}) \to f(x_0^1, \ldots, x_0^{i_1-1}, x^{i_1}, x_0^{i_1+1}, \ldots, x_0^{i_p-1}, x^{i_p}, x_0^{i_p+1}, \ldots, x_0^s)$$

is analytic in a neighborhood of $(x_0^{i_1}, \ldots, x_0^{i_p})$ in $\mathbf{R}^{n_{i_1}} \times \ldots \times \mathbf{R}^{n_{i_p}}$.

Given a function $f : \Omega \to \mathbf{C}$ p-separately analytic in Ω, we put

$$A := \{x \in \Omega;\ f \text{ is analytic in a neighborhood of } x\}.$$

The set $S := \Omega \setminus A$ is called the *singular set* of f.

The aim of this paper is to prove the following two theorems.

THEOREM 1. *Let p be an integer with $s/2 \leq p < s$ and let f be a p-separately analytic function in an open set Ω in $\mathbf{R}^{n_1} \times \ldots \times \mathbf{R}^{n_s}$. Then for every sequence $1 \leq j_1 < \ldots < j_q \leq s$, where $q := s - p$, the projection $S_{j_1 \ldots j_q}$ of the singular set S of f on $\mathbf{R}^{n_{j_1}} \times \ldots \times \mathbf{R}^{n_{j_q}}$ is pluripolar in $\mathbf{C}^{n_{j_1}} \times \ldots \times \mathbf{C}^{n_{j_q}}$.*

THEOREM 2. *Given any fixed integer p with $1 \leq p < s$, let S be a closed subset of an open set Ω in $\mathbf{R}^{n_1} \times \ldots \times \mathbf{R}^{n_s}$ such that for every sequence $1 \leq j_1 < \ldots < j_q \leq s$, where $q := s - p$, the projection $S_{j_1 \ldots j_q}$ is pluripolar in $\mathbf{C}^{n_{j_1}} \times \ldots \times \mathbf{C}^{n_{j_q}}$. Then there is a p-separately analytic function $f : \Omega \to \mathbf{C}$ such that S is its singular set.*

Taking $s = 2$, $m = n_1$, $n = n_2$, $p = 1$, one gets the following

COROLLARY 1. *Let S be a closed subset of an open set Ω in $\mathbf{R}^m \times \mathbf{R}^n$. Let S_1 and S_2 be the projections of S on $\mathbf{R}^m$ and $\mathbf{R}^n$, respectively. Then S_1 and S_2 are pluripolar in $\mathbf{C}^m$ and $\mathbf{C}^n$, respectively, if and only if S is the singular set of a separately analytic function $f : \Omega \ni (x, u) \to f(x, u) \in \mathbf{C}$, where $x \in \mathbf{R}^m$, $u \in \mathbf{R}^n$.*

COROLLARY 2. *Let S be a closed subset of an open set in $\mathbf{R}^n$ $(n \geq 2)$. Then S is a singular set of an $(n-1)$-separately analytic function $f : \Omega \to \mathbf{C}$ if and only if for each $j = 1, \ldots, n$ the projection S_j of S on the real coordinate line x_j is polar as a subset of the complex z_j-plane, where $z_j = x_j + iy_j$.*

If $m = 1$, $n = 1$, Corollary 1 is equivalent to Saint Raymond's result [4]. We do not know whether Theorem 1 remains true for $1 \leq p < s/2$.

EXAMPLE. It is clear that if f is p-separately analytic, then it is q-separately analytic, where $1 \leq q < p$. On the other hand, the function

$$f(x^1, \ldots, x^s) := |x^1|^2 \ldots |x^s|^2 \exp\Big(-1 \Big/ \sum_{i=1}^{p+1} |x^i|^2\Big), \tag{1}$$

where $1 \leq p < s$, is p-separately analytic in $\mathbf{R}^{n_1} \times \ldots \times \mathbf{R}^{n_s}$ but it is not $(p+1)$-separately analytic. Here $|x^i|$ denotes the euclidean norm of the vector x^i in the space $\mathbf{R}^{n_i}$. The singular set S of the function (1) is given by

$$S = \{(x^1, \ldots, x^s) \in \mathbf{R}^{n_1} \times \ldots \times \mathbf{R}^{n_s};\ |x^i| = 0 \quad (i = 1, \ldots, p+1)\}.$$

We shall need the following known results:

THEOREM 0.1 (Bremermann [2]). *If u is a plurisubharmonic (plsh) function in a domain of holomorphy D in $\mathbf{C}^n$ then there exists a sequence $\{f_j\}$ of holomorphic functions in D such that the functions $(1/j)\log|f_j|$ are locally uniformly upper bounded in D and $u(z) = v^*(z)$, where $v(z) := \limsup_{j\to\infty}(1/j)\log|f_j(z)|$ and $v^*(z) := \limsup_{a\to z} v(a)$.*

THEOREM 0.2 (Hartogs Lemma [3]). *Let $\{u_j\}$ be a locally uniformly upper bounded sequence of plurisubharmonic functions in a domain D in $\mathbf{C}^n$. If $\limsup_{j\to\infty} u_j \leq m$ in D, then for every compact subset K of D and for every $\varepsilon > 0$ there is $j_0 = j_0(K, \varepsilon)$ such that $u_j(z) < m + \varepsilon$ for $j > j_0$ and $z \in K$.*

THEOREM 0.3 (Bedford–Taylor theorem on negligible sets [1]). *Let $\{u_j\}$ be a locally uniformly upper bounded sequence of plsh functions in a domain D in $\mathbf{C}^n$. Then the sets*

$$\{z \in D; u(z) := \limsup_{j\to\infty} u_j(z) < u^*(z)\},$$
$$\{z \in D; v(z) := \sup_j u_j(z) < v^*(z)\}$$

are pluripolar (shortly plp).

THEOREM 0.4 [6]. *Let B be a fixed bounded domain in $\mathbf{C}^n$, e.g. a unit ball. A compact subset K of $\mathbf{C}^n$ is pluripolar if and only if $\alpha(K) = 0$, where*
$\alpha(K) := \inf_j \alpha_j^{1/j}(K)$,

$\alpha_j(K) := \inf\{\|p\|_K;\ p$ *is a polynomial on* $\mathbf{C}^n$ *of degree* $\leq j$ *such that* $\|p\|_B = 1\}$.

Let D_j be a simply connected domain in the z_j-plane symmetric with respect to the real x_j-axis. Let $[a_j, b_j]$ be a compact interval of the real axis contained in D_j $(j = 1, \ldots, m)$. Let G_k be a simply connected domain in the w_k-plane symmetric with respect to the u_k-axis and let $[c_k, d_k]$ be a compact interval of the real axis contained in G_k. Put

$$X := [a,b] \times G \cup D \times [c,d],$$

where $[a,b] := [a_1, b_1] \times \ldots \times [a_m, b_m]$, $D := D_1 \times \ldots \times D_m$, $[c,d] := [c_1, d_1] \times \ldots \times [c_n, d_n]$, $G := G_1 \times \ldots \times G_n$. Let $h(z_j, [a_j, b_j], D_j)$ be the continuous function on D_j, harmonic in $D_j \setminus [a_j, b_j]$, equal to 0 on $[a_j, b_j]$ and to 1 on ∂D_j. It is clear that

$$\widetilde{X} := \{(z,w) \in D \times G;\ \max_{1 \leq j \leq m} h(z_j, [a_j, b_j], D_j) + \max_{1 \leq k \leq n} h(w_k, [c_k, d_k], G_k) < 1\}$$

is an open neighborhood of X.

We say that a function $f : X \to \mathbf{C}$ is *separately holomorphic* on X if

(i) for every x in $[a,b]$ the function $f(x, \cdot)$ is holomorphic in G;
(ii) for every u in $[c,d]$ the function $f(\cdot, u)$ is holomorphic in D.

The following theorem can be easily deduced from Theorem 2a of [5].

THEOREM 0.5. *There is a family* $\{\Phi_{\alpha\beta} : (\alpha, \beta) \in \mathbf{Z}_+^m \times \mathbf{Z}_+^n\}$ *of holomorphic functions in* $D \times G$ *such that every function* f *bounded* [1] *and separately holomorphic on* X *can be expanded in a series*

$$f(z,w) = \sum c_{\alpha\beta} \Phi_{\alpha\beta}(z,w), \qquad (z,w) \in X. \tag{2}$$

Moreover, the series (2) *is locally uniformly convergent in* $\widetilde{X}$, *so that its sum* $\widetilde{f}$ *gives a holomorphic extension of* f *from* X *to* $\widetilde{X}$. *In particular, every function* f *bounded and separately holomorphic on* X *is real analytic in a neighborhood of* $[a,b] \times (\mathbf{R}^n \cap G) \cup (\mathbf{R}^m \cap D) \times [c,d]$ *in* $\mathbf{R}^m \times \mathbf{R}^n$.

1. Proof of Theorem 1.

First we shall prove

THEOREM 1'. *Let* $f : \Omega \ni (x,u) \to f(x,u) \in \mathbf{C}$ *be a separately analytic function of two vector variables* $x \in \mathbf{R}^m$ *and* $u \in \mathbf{R}^n$, *where* Ω *is an open subset of* $\mathbf{R}^m \times \mathbf{R}^n$. *Let* S *be the singular set of* f *and let* S_1 *and* S_2 *be its projections on* $\mathbf{R}^m$ *and* $\mathbf{R}^n$, *respectively. Then* S_1 *is plp in* $\mathbf{C}^m$ *and* S_2 *is plp in* $\mathbf{C}^n$.

(1) This assumption is superfluous [5], but it makes the proof much easier.

Proof. Fix $I = [a,b] = [a_1,b_1] \times \ldots \times [a_m,b_m]$, $J = [c,d] = [c_1,d_1] \times \ldots \times [c_n,d_n]$ such that $I \times J \subset \Omega$. It is sufficient to show that $S_1 \cap I$ and $S_2 \cap J$ are pluripolar.

For every integer $k \geq 1$ the set

$$J^k := \{w \in \mathbf{C}^n;\ \max_{1 \leq j \leq n} \operatorname{dist}(w_j, [c_j, d_j]) < 1/k\}$$

is an open neighborhood of J such that $J^{k+1} \subset J^k$, $J = \bigcap_{k=1}^\infty J^k$. The set

$$E_k := \{x \in I; f(x,\cdot) \text{ is holomorphic in } J^k \text{ and } \sup_{w \in J^k} |f(x,w)| \leq k\}$$

is closed, $E_k \subset E_{k+1}$ and $I = \bigcup_{k=1}^\infty E_k$. By the Baire property of $\mathbf{R}^m$ the set $U := \bigcup_{k=1}^\infty \mathring{E}_k$ is open and dense in I. Similarly we define an open dense subset V of J. We claim that

1° *The set $(I \times V) \cup (U \times J)$ is contained in the domain of analyticity A of f.*

We shall prove that $I \times V \subset A$. The proof of the inclusion $U \times J \subset A$ is analogous. Fix (x_0, u_0) in $I \times V$ and let $[\gamma, \delta]$ be an n-dimensional closed interval such that $u_0 \in [\gamma,\delta] \subset V$. Then $|f(z,u)| \leq k$ for all (z,u) in $I^k \times [\gamma,\delta]$ and $f(\cdot,u)$ is holomorphic in I^k for every fixed u in $[\gamma,\delta]$ and for all $k > k_0$.

Let $[\alpha,\beta]$ be an m-dimensional closed interval contained in U. Then $|f(x,w)| \leq k$ for all (x,w) in $[\alpha,\beta] \times J^k$ and $f(x,\cdot)$ is holomorphic in J^k for each fixed x in $[\alpha,\beta]$ and $k > k_1$. Hence if $k > \max(k_0,k_1)$ then f is separately holomorphic and bounded by k on $[\alpha,\beta] \times J^k \cup I^k \times [\gamma,\delta]$. By Theorem 0.5, f is real analytic in a neighborhood of $I \times [\gamma,\delta]$, in particular $(x_0,u_0) \in A$.

We shall now prove that $S_1 \cap I$ and $S_2 \cap J$ are pluripolar. It is enough to show that $S_1 \cap I$ is plp. Define

$$Q(x,u) := \sup_\alpha \left| \frac{1}{\alpha!} \left(\frac{\partial}{\partial u}\right)^\alpha f(x,u) \right|^{1/|\alpha|}, \qquad (x,u) \in \Omega, \tag{1.1}$$

the supremum being taken over all α in $\mathbf{Z}_+^n$. Then

$$f(x,u+w) = \sum \frac{1}{\alpha!} \left(\frac{\partial}{\partial u}\right)^\alpha f(x,u) w^\alpha \quad \text{and} \tag{1.2}$$

$$|f(x,u+w)| \leq (1 - |w| Q(x,u))^{-n}$$

if $|w| < 1/Q(x,u)$, $(x,u) \in \Omega$, $w \in \mathbf{C}^n$.

We claim that

2° *For each fixed u in V, the function $Q(\cdot,u)$ is quasi-almost everywhere (q.a.e.) continuous on I, i.e. $Q(\cdot,u)$ is continuous on $I \setminus E$, where E is a plp set in $\mathbf{C}^m$.*

Indeed, f being analytic on $I \times V$, we may assume f is defined and holomorphic on $D \times B$, where D is a domain in $\mathbf{C}^m$ with $I \subset D$, and $B = B(u, r)$ is a ball in $\mathbf{C}^n$ with center u and radius $r > 0$. By the Bedford–Taylor Theorem 0.3 the set

$$N := \left\{ z \in D; \varphi(z) := \sup_{\alpha} \left| \frac{1}{\alpha!} \left(\frac{\partial}{\partial u} \right)^{\alpha} f(z, u) \right|^{1/|\alpha|} < \varphi^*(z) \right\}$$

is pluripolar. The function φ is lower semicontinuous as an upper envelope of continuous functions. Therefore N is identical with the set of discontinuity points of φ. In particular, the function $x \to Q(x, u) \equiv \varphi(x)$ is q.a.e. continuous on I.

Now we shall prove:

3° *If $(x_0, u_0) \in S$ then there exists $r > 0$ such that for every u in $V \cap B(u_0, r)$ the function $Q(\cdot, u)$ is discontinuous at x_0.*

Suppose on the contrary that for every $r > 0$ there is u in $V \cap B(u_0, r)$ such that $Q(\cdot, u)$ is continuous at x_0. Put $R := 1/Q(x_0, u_0)$. Take u in $V \cap B(u_0, R/4)$ such that $Q(\cdot, u)$ is continuous at x_0. It is clear that $3Q(x_0, u) < 4Q(x_0, u_0)$. By the continuity of $Q(\cdot, u)$ at x_0 there is an interval $[\alpha, \beta]$ in $\mathbf{R}^m$ with $x_0 \in [\alpha, \beta] \subset I$ such that $3Q(x, u) < 4Q(x_0, u_0)$ for every x in $[\alpha, \beta]$. Hence $f(x, \cdot)$ is holomorphic in the ball $B(u_0, R/2)$ for every x in $[\alpha, \beta]$. Moreover, by (1.2), f is bounded on $[\alpha, \beta] \times B(u_0, R/3)$. On the other hand, by the argument used in the proof of 1° one can find an interval $[\gamma, \delta]$ in $J \cap B(u_0, R/3)$ and a domain D in $\mathbf{C}^m$ such that $I \subset D$ and for every u in $[\gamma, \delta]$ the function $f(\cdot, u)$ is holomorphic in D. Moreover, f is bounded in $D \times [\gamma, \delta]$. So f is bounded and separately holomorphic on

$$[\alpha, \beta] \times B(u_0, R/3) \cup D \times [\gamma, \delta],$$

which shows (by Theorem 0.5) that $(x_0, u_0) \in A$. The proof of 3° is complete.

Now we are ready to prove that $S_1 \cap I$ is pluripolar. Let W be a countable dense subset of J. By 3°, $S_1 \cap I \subset \bigcup_{u \in V \cap W} E_u$, where E_u is the set of discontinuity points of $Q(\cdot, u)$ in I. By 2°, E_u is pluripolar in $\mathbf{C}^m$. Therefore $S_1 \cap I$ is pluripolar, as a subset of a countable union of pluripolar sets. The proof of Theorem 1′ is complete.

It remains to show that Theorem 1′ implies Theorem 1. Given fixed sequences $1 \le i_1 < \ldots < i_p \le s$ and $1 \le j_1 < \ldots < j_q \le s$ such that $\{1, \ldots, s\} = \{i_1, \ldots, i_p\} \cup \{j_1, \ldots, j_q\}$, $p + q = s$, put

$$\begin{aligned} \Omega^* := \{ (x, u) \in (\mathbf{R}^{n_{i_1}} \times \ldots \times \mathbf{R}^{n_{i_p}}) \times (\mathbf{R}^{n_{j_1}} \times \ldots \times \mathbf{R}^{n_{j_q}}); \\ x = (x^{i_1}, \ldots, x^{i_p}),\ u = (x^{j_1}, \ldots, x^{j_q}),\ (x^1, \ldots, x^s) \in \Omega \}. \end{aligned}$$

Then by the inequalities $s/2 \le p < s$ the function

$$f^*(x,u) := f(x^1,\dots,x^s), \qquad (x,u) \in \Omega^*,$$

is separately analytic in Ω^*. By Theorem 1′ the projection $S_{j_1\dots j_q}$ of its singular set S on $\mathbf{R}^{n_{j_1}} \times \dots \times \mathbf{R}^{n_{j_q}}$ is pluripolar in $\mathbf{C}^{n_{j_1}} \times \dots \times \mathbf{C}^{n_{j_q}}$.

2. Auxiliary lemmas. The proof of Theorem 2 will be based on several lemmas.

LEMMA 2.1 [4]. *If Ω' is a bounded open subset of $\mathbf{R}^N$, then there is a function ψ plurisubharmonic in $\mathbf{C}^N$ such that*

1° $\psi(x) < 0$ *for x in Ω',*
2° $\psi(x) = 0$ *for x in $\mathbf{R}^N \setminus \Omega'$.*

Proof. The function

$$\varphi(\lambda) := \begin{cases} 0 & \text{if } \operatorname{Re}\lambda \ge 0, \\ \operatorname{Re}(\lambda^3) & \text{if } \operatorname{Re}\lambda < 0, \end{cases}$$

is subharmonic (because it is harmonic) at all points λ with $\operatorname{Re}\lambda \neq 0$. If $\lambda = i\beta$, $\beta \in \mathbf{R}$, then $\varphi(i\beta) = 0 < (2\pi)^{-1}\int_0^{2\pi} \varphi(i\beta + re^{it})\,dt = (2\pi)^{-1}\int_{\pi/2}^{3\pi/2} \operatorname{Re}((i\beta + re^{it})^3)\,dt = \frac{2}{3}r^3$, $r > 0$, which implies that φ is subharmonic everywhere on the complex plane $\mathbf{C}$.

Let $\{x_k = (x_{k1},\dots,x_{kN}); k = 1,2,\dots\}$ be a countable dense subset of Ω' and let $r_k := \operatorname{dist}(x_k, \partial\Omega')$. Then the function

$$\psi(z) := \sum_{k=1}^{\infty} 2^{-k} \varphi\Big(\sum_{j=1}^{N} (z_j - x_{k_j})^2 - r_k^2\Big), \qquad z \in \mathbf{C}^N,$$

is plurisubharmonic in $\mathbf{C}^N$ and satisfies 1° and 2°. Since the series is locally uniformly convergent in $\mathbf{C}^N$, the function is continuous.

LEMMA 2.2. *Let E be an F_σ pluripolar subset of I^N, where $I := \{t \in \mathbf{R}; -1 < t < 1\}$. Then there exists an increasing sequence $\{m_l\}$ of positive integers and a sequence $\{P_l\}$ of polynomials of N complex variables such that*

(a) $|P_l(z)| \le 1$ *in Δ^N, where Δ^N is the unit polydisc in $\mathbf{C}^N$;*
(b) $\lim_{l\to\infty} |P_l(z)|^{1/m_l} = 0$ *on E;*
(c) $\lim_{l\to\infty} |P_l(z)|^{1/m_l} = 1$ *q.a.e. in $\mathbf{C}^N$.*

Proof. Let K_i be a compact subset of E with $K_i \subset K_{i+1}$, $E = \bigcup_{i=1}^{\infty} K_i$. By Theorem 0.4 for every i there is a polynomial p_i such that

$$\|p_i\|_{\Delta^N} = 1 \quad \text{and} \quad d_i^{-1} \log|p_i(z)| \le -2^i \quad \text{on} \quad K_i, \tag{2.1}$$

where $d_i := \deg p_i \geq 1$. Put

$$u(z) := \sum_{i=1}^{\infty} 2^{-i} d_i^{-1} \log |p_i(z)| .$$

Then u is a plurisubharmonic function in $\mathbf{C}^N$ such that $u(z) \leq \log^+ |z|$ in $\mathbf{C}^N$ and $u(z) = -\infty$ on E. Observe that

$$u(z) = \lim_{s \to \infty} \log \prod_{i=1}^{s} |p_i(z)|^{1/2^i d_i} = \lim_{s \to \infty} \mu_s^{-1} \log |Q_s(z)| ,$$

where $\mu_s := \prod_{i=1}^{s} 2^i d_i$ and $Q_s(z) := \prod_{i=1}^{s} p_i(z)^{\mu_s/2^i d_i}$.

It follows from (2.1) that

$$\mu_s^{-1} \log |Q_s(z)| \leq \log \prod_{i=l+1}^{s} |p_i(z)|^{1/2^i d_i} \leq -(s-l) , \qquad z \in K_l ,\ s > l .$$

Given $l \geq 1$ take s_l so large that $s_l - l > l^2$. Then $\mu_{s_l}^{-1} \log |Q_{s_l}(z)| \leq -l^2$ on K_l. It is now clear that $m_l := l\mu_{s_l}$ and $P_l := Q_{s_l}$ ($l \geq 1$) are the required sequences.

Let $\mathcal{P}(\mathbf{N})$ be the space of all subsets A of $\mathbf{N}$ (natural numbers). $\mathcal{P}(\mathbf{N})$ can be identified with the space $\{0,1\}^{\mathbf{N}}$ endowed with the product topology. Therefore $\mathcal{P}(\mathbf{N})$ is a compact metrizable space.

LEMMA 2.3 (Saint Raymond [4]). *Let $\Omega \subset \Omega_1$ be open subsets of $\mathbf{C}^N$ and let a be a fixed point of $\overline{\Omega} \cap \Omega_1$. Let $\{g_k\}$ be a sequence of holomorphic functions in Ω_1 such that*

$$\sum_{k=1}^{\infty} |g_k(z)| < +\infty \qquad \textit{for every } z \textit{ in } \Omega.$$

Then either the series $\sum_{k=1}^{\infty} g_k$ is normally convergent in a neighborhood of a, or there exists a rare (i.e. of the first Baire category) subset M of $\mathcal{P}(\mathbf{N})$ such that if $A \in \mathcal{P}(\mathbf{N}) \setminus M$ then the function $f_A := \sum_{k \in A} g_k$ cannot be continued to a holomorphic function in a neighborhood of a.

3. Proof of Theorem 2. Without loss of generality we may assume that Ω is contained in $I^{n_1} \times \ldots \times I^{n_s} \equiv I^N$, where $I := \{t \in \mathbf{R}; -1 < t < 1\}$, $N := n_1 + \ldots + n_s$. Indeed, by the bianalytic mapping

$$I^N \ni \xi \to \left(\tan \frac{\pi \xi_1}{2}, \ldots, \tan \frac{\pi \xi_N}{2} \right) \in \mathbf{R}^N = \mathbf{R}^{n_1} \times \ldots \times \mathbf{R}^{n_s} ,$$

the general situation is reduced to the special one.

Given $1 \leq j_1 < \ldots < j_q \leq s$, one can find (by Lemma 2.2) a sequence of positive integers $(m_{lj_1 \ldots j_q})_{l \geq 1}$ and a sequence of polynomials $(P_{lj_1 \ldots j_q})_{l \geq 1}$ in $(z^{j_1}, \ldots, z^{j_q}) \in \mathbf{C}^{n_{j_1}} \times \ldots \times \mathbf{C}^{n_{j_q}}$ such that

(i) $|P_{lj_1\dots j_q}| \le 1$ on $\Delta^{n_{j_1}} \times \dots \times \Delta^{n_{j_q}}$,
(ii) $\lim_{l\to\infty} m_{lj_1\dots j_q}^{-1} \log|P_{lj_1\dots j_q}| = -\infty$ on $S_{j_1\dots j_q}$,
(iii) $\lim_{l\to\infty} m_{lj_1\dots j_q}^{-1} \log|P_{lj_1\dots j_q}| = 0$ q.a.e. in $\mathbf{C}^N$.

By Lemma 2.1 there exists a function ψ plurisubharmonic in $\mathbf{C}^N$ such that

(iv) $\psi < 0$ on $\Omega' := \Omega \setminus S$,
(v) $\psi = 0$ on $\mathbf{R}^N \setminus \Omega'$.

Let $\{\psi_\nu\}$ be a sequence of entire functions (given by Theorem 0.1) such that the functions $(1/\nu)\log|\psi_\nu|$ are locally uniformly upper bounded in $\mathbf{C}^N$ and

(vi) $\psi = (\limsup_{\nu\to\infty}(1/\nu)\log|\psi_\nu|)^*$ in $\mathbf{C}^N$.

Put

$$g_k(z) := \psi_k(z)^{m_k} \prod_{1\le j_1<\dots<j_q\le s} P_{kj_1\dots j_q}(z^{j_1},\dots,z^{j_q})^{\alpha_{kj_1\dots j_q}}, \quad z \in \mathbf{C}^N,$$

$$m_k := \prod_{1\le j_1<\dots<j_q\le s} m_{kj_1\dots j_q}, \quad \alpha_{kj_1\dots j_q} := km_k/m_{kj_1\dots j_q}.$$

We shall show that the required function f is given by

$$f = \sum_{k\in A} g_k$$

where A is a subset of $\mathbf{N}$.

Indeed, by (i) the functions $(km_k)^{-1}\log|g_k|$ are locally uniformly upper bounded in the polydisc Δ^N, and

$$\limsup_{k\to\infty}(km_k)^{-1}\log|g_k(z)| \le \limsup_{k\to\infty} k^{-1}\log|\psi_k(z)| \quad \text{in } \Delta^N.$$

Therefore by (iii) and (vi)

$$\psi(z) = (\limsup_{k\to\infty}(km_k)^{-1}\log|g_k|)^*(z) \quad \text{in } \Delta^N. \tag{3.1}$$

The set $\widetilde{\Omega}' := \{z \in \Delta^N; \psi(z) < 0\}$ is an open neighborhood of $\widetilde{\Omega}'$. We claim the sequence $\{g_k\}$ has the following properties (a), (b) and (c).

(a) *The series $\sum_{k=1}^\infty g_k$ is locally normally convergent in $\widetilde{\Omega}'$. More exactly, for every fixed z_0 in $\widetilde{\Omega}'$ there are a compact neighborhood V of z_0 and real numbers $0 < \theta < 1$ and k_0 such that*

$$|g_k(z)| \le \theta^{km_k}, \qquad z \in V,\ k > k_0. \tag{3.2}$$

(b) *For every fixed* $(z_0^{j_1},\dots,z_0^{j_q})$ *in* $S_{j_1\dots,j_q}$ *the series*

$$\sum_{k=1}^{\infty} g_k(z^1,\dots,z^{j_1-1},z_0^{j_1},z^{j_1+1},\dots,z^{j_q-1},z_0^{j_q},z^{j_q+1},\dots,z^s)$$

is locally normally convergent in $\Delta^{n_{i_1}}\times\dots\times\Delta^{n_{i_p}}$.

(c) *Let* z_0 *be an arbitrary point of* S. *Then every neighborhood* U *of* z_0 *contains a point* z' *such that the series* $\sum_{k=1}^{\infty} g_k(z')$ *is divergent.*

Proof of (a). Given a compact neighborhood U of a point $z_0 \in \widetilde{\Omega}'$ with $U \subset \widetilde{\Omega}'$ we have $\theta' := \exp\sup_U \psi < 1$. Take θ with $\theta' < \theta < 1$ and let V be a compact neighborhood of z_0 with $V \subset U$. Then by the Hartogs Lemma (Theorem 0.2) one gets the required inequalities (3.2).

Proof of (b). This follows from (i), (ii) and from the locally uniform upper boundedness of the sequence $k^{-1}\log|\psi_k|$.

Proof of (c). By (iv) and (v), given any neighborhood U of a point z_0 in S, the set $\{z \in U; \psi(z) > 0\}$ is not pluripolar. Therefore by (3.1) and the Bedford–Taylor theorem on negligible sets $\limsup_{k\to\infty}(km_k)^{-1}\log|g_k(z)| > 0$ on a nonpluripolar subset E of U. It is clear that $\sum_{k=1}^{\infty} g_k$ diverges at each point of E.

It follows from (a) and (b) that for every subset A of $\mathbf{N}$ the series

$$f_A(z) := \sum_{k\in A} g_k(z)$$

represents a p-separately analytic function in Ω which is jointly analytic in $\Omega' := \Omega\setminus S$. By (c) and by Saint Raymond's Lemma 2.3 there is $A \in \mathbf{N}$ such that each point of S is singular for f_A. The proof of Theorem 2 is complete.

PROBLEMS. 1. The proof of Theorem 2 based on Saint Raymond's Lemma 2.3 gives only the existence of the function f. It would be interesting to give an effective construction of the required function.

2. Does there exist a $\mathcal{C}^\infty$ (resp. continuous) p-separately analytic function $f : \Omega \to \mathbf{C}$ such that S is its singular set?

3. Characterize the singular sets of $\mathcal{C}^\infty$ (resp. continuous) p-separately analytic functions.

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