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AN EXAMPLE OF 2-DIMENSIONAL QUASI-HOMEOMORPHIC SETS

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1. Introduction. We shall consider only compacta, i.e. compact metric spaces. As defined by Mardešić and Segal in [7], a compactum X is like to a compactum Y – or, simply, Y-like – if for every $\varepsilon > 0$ there is an ε-mapping of X onto Y. A map $f: X \to Y$ is said to be an ε-mapping if $\operatorname{diam} f^{-1}(y) < \varepsilon$ for every $y \in f(X)$. Two compacta X and Y are quasi-homeomorphic if X is Y-like and Y is X-like (this notion was given by Kuratowski and Ulam in 1933 in the paper [6]).

In [3] K. Borsuk found a famous example of a 3-dimensional acyclic locally connected continuum, which admits a homeomorphism onto itself without any fixed point. He next used this example in the paper [4] to answer in the negative the question of Kuratowski and Ulam whether the fixed point property is a quasi-homeomorphism invariant. Namely, he proved that his example mentioned above is quasi-homeomorphic with the 3-cell. In the same paper it is proved that the fixed point-property is a quasi-homeomorphism invariant in the class of compact ANR's (and therefore the counter-example cannot be an ANR). In [1] (cf. also [2]) R. H. Bing has found a 2-dimensional version of the Borsuk's example [3], i.e. an example of a 2-dimensional acyclic locally connected continuum being the intersection of a sequence of 3-cells, which admits a homeomorphism onto itself without any fixed point. It is easy to see (cf. section 2), that the Bing's example is Z-like, where Z is a 2-dimensional AR-set obtained as the union of two disks D_1 and D_2 sticked together in such a way that some arcs $I_1 \subset \mathring{D}_1$ and $I_2 \subset \mathring{D}_2$ are identified by a homeomorphism. (Here, as usual, $\mathring{D}$ denotes the interior of the disk D.) However, it can probably be proved that the Bing's example is not quasi-homeomorphic with any ANR (which, of course, must be 2-dimensional). By modifying the Bing's example, we prove in this note that there is another 2-dimensional acyclic locally connected continuum without the fixed point-property, which is quasi-homeomorphic with a 2-dimensional AR-set.

It is easy to see that a 1-dimensional version of the Borsuk's example [3] cannot exist, because any compactum quasi-homeomorphic with a local dendron is a local dendron. It has been proved by Lê Xuân Bĩnh [5] that a compactum is quasi-homeomorphic with a disk iff it is a 2-dimensional AR-set embeddable into the plane E^2, and therefore there cannot exist an example of the considered sort, which is quasi-homeomorphic with a disk. However, it is not known if there is such an example, which is quasi-homeomorphic with a 2-dimensional polyhedron. The above-mentioned result of Lê Xuân Biñh suggests that the answer to this question is probably negative. On the other hand, since our example does not admit a homeomorphism onto itself without fixed points, it is neither known if there is a compactum quasi-homeomorphic with a 2-dimensional AR-set, which admits such a homeomorphism(¹).

2. The example. First, let us describe as in [2] the Bing's example mentioned above. It will be called X_0. Consider first a cylinder given by the equation $x^2+y^2=4$ with two bases B^- and B^+ lying respectively on the planes $z=-\frac{1}{2}$ and $z=\frac{1}{2}$. Next, form two cones C^- and C^+, whose bases respectively are B^- and B^+ and whose vertices respectively are the points $v^+ = (2, 0, \frac{1}{2})$ and $v^- = (-2, 0, -\frac{1}{2})$. Then, we construct the set X_0 from the set $C^- \cup C^+$ by the rotation of the intersection of this set with any horizontal plane $z = z_0$, where $-\frac{1}{2} < z_0 < \frac{1}{2}$, about the z-axis by the angle $\tan \pi z_0$. Thus $X_0 = \tilde{C}^+ \cup \tilde{C}^-$, where $\tilde{C}^+$ ($\tilde{C}^-$) is the image of $C^+ \setminus \{v^-\}$ ($C^- \setminus \{v^+\}$) under this function of $C^+ \cup C^-$ onto X_0 (the function is the identity on $B^+ \cup B^-$). For each θ, where $0 < \theta < 2\pi$, there is a homeomorphism h_θ of the set X_0 onto itself without any fixed point. This homeomorphism h_θ rotates the bases B^+ and B^- about the z-axis by the angle θ and on the intersections of the set X_0 with other horizontal planes it acts by the same rotation, suitable lift and stretch (i.e., contracting of one circle and expanding of the other so that they agree with the lifted ones). Under this homeomorphism the tangent spiral S of the rotated cones $\tilde{C}^+$ and $\tilde{C}^-$ is moved onto itself by these rotation and lift.

Now, we shall describe the AR-set Y and after that the compactum X without the fixed-point property, which is quasi-homeomorphic with Y. The set Y will be the union of AR-sets Y_n, $n = 0, 1, 2, \ldots$, constructed as follows:

Let I denote the tangent segment of the cones C^+ and C^-, i.e. the segment joining the vertices v^+ and v^-. Find a sequence $I_0, I_1^+, I_1^-, I_2^+, I_2^-, \ldots$ of disjoint segments lying on I such that I_j^+ and I_j^- are symmetric with respect to the center 0 of I, diam $I_j^+ =$ diam $I_j^- < 1/j$ for $j \neq 0$, the segments I_j^+ (I_j^-), $j = 1, 2, \ldots$, converge to v^+ (v^-). Assume that the end-points w^+ and w^- of I_0 are also symmetric with respect to 0 with w^- having the negative z-coordinate and that the ordering of the segments I_j^+'s on I agrees with the ordering of their indices. Moreover, find another sequence $J_1^+, J_1^-, J_2^+, J_2^-, \ldots$ of segments such that $J_j^+ \subset \mathring{I}_j^+$, $J_j^- \subset \mathring{I}_j^-$, $j = 1, 2, \ldots$

(¹) See T. Maćkowiak, *A continuum without the fixed point property which is quasi-homeomorphic with an AR-set*, this fascicle, p. 79–80. [Note of the Editors] (cf. Fig. 1). We shall assume that $\alpha_n^\varepsilon([w^+, w^-]) \supset \beta_n^\varepsilon(\bigcup_{j<n} I_j^+ \cup I_j^-)$, where α_n^ε (β_n^ε) is a homothety of I onto I_n^ε (J_n^ε) and $\varepsilon = \mp$.

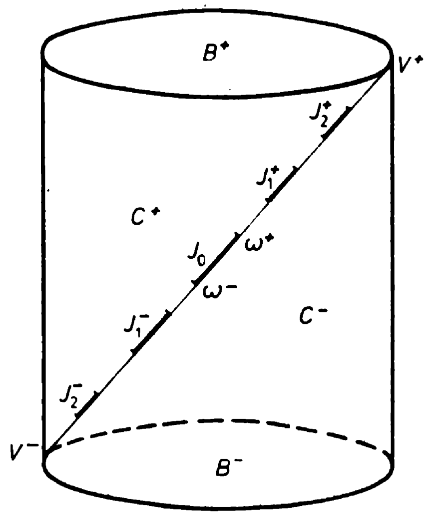

Fig. 1. (Added in proof: J_0 should be read I_0.)

Now, define Y_0 as the union of two cones D^- with the base B^- and the vertex w^+ and D^+ with the base B^+ and the vertex w^-. Notice that Y_0 is topologically the set Z such that X_0 is Z-like, mentioned in the Introduction.

Next, we shall construct the set Y_1 as the union $Y_0 \cup \tilde{Y}_0^+ \cup \tilde{Y}_0^-$, where the sets $\tilde{Y}_0^+$ and $\tilde{Y}_0^-$ are constructed as follows: Construct two homothetic images Y_{00}^+ and Y_0^+ of Y_0 such that $\operatorname{diam} Y_{00}^+ = \operatorname{diam} I_1^+$, $\operatorname{diam} Y_0^+ = \operatorname{diam} J_1^+$ and let L_{00}^+, L_0^+ denote the respective images of the segment I, where $L_{00}^+ \subset Y_{00}^+$, $L_0^+ \subset Y_0^+$. Then identify the segment L_{00}^+ with the segment $I_1^+ \subset Y_0$ and the segment L_0^+ with the segment $J_1^+ \subset Y_0$ by linear homeomorphisms. Now, the set obtained from the disjoint union $Y_{00}^+ \cup Y_0^+$ under these identifications is $\tilde{Y}_0^+$. The second set $\tilde{Y}_0^-$ is constructed in the symmetric manner.

If the set Y_n, $n \geqslant 1$, is defined, we define the set $Y_{n+1} = Y_n \cup \tilde{Y}_n^+ \cup \tilde{Y}_n^-$ in a similar way: Construct a homothetic image Y_{0n}^+ of Y_0 such that $\operatorname{diam} Y_{0n}^+ = \operatorname{diam} I_{n+1}^+$ and a homothetic image Y_n^+ of Y_n such that $\operatorname{diam} Y_n^+ = \operatorname{diam} J_{n+1}^+$. Then define $\tilde{Y}_n^+$ as the set obtained from the disjoint union $Y_{0n}^+ \cup Y_n^+$ under the identifications of the segments L_{0n}^+ and $I_{n+1}^+ \subset Y_0 \subset Y_n$ as well as the segments L_n^+ and J_{n+1}^+ by linear homeomorphisms. (Here, $L_{0n}^+ \subset Y_{0n}^+$ and $L_n^+ \subset Y_n^+$ are the respective homothetic images of the segment I.) The sets Y_{0n}^+ and Y_n^+ will be non-distinguished from their images under these identifications. The second set $\tilde{Y}_n^-$ is constructed in the symmetric way (cf. Fig. 2).

It is obvious that the set Y being the union of all Y_n's, $n = 0, 1, 2, \ldots$ is an AR-set.

Finally, we construct the above-mentioned compactum X. Consider the tangent spiral S contained in X_0 and construct a sequence K_1^+, K_1^-, K_2^+, K_2^-, ... of disjoint arcs contained in S with diam $K_i^+ =$ diam $K_i^- < 1/i$ for $i = 1, 2, \ldots$, which are placed on S similarly as the segments I_1^+, I_1^-, I_2^+, I_2^-, ... are placed on I. We can assume that the arcs K_i^+ (K_i^-) converge to a point belonging to the base $B^+ \subset X_0$ ($B^- \subset X_0$). Then consider the disjoint union of X_0 and of all sets $\tilde{Y}_n^+$, $\tilde{Y}_n^-$, where $n = 0, 1, 2, \ldots$, constructed

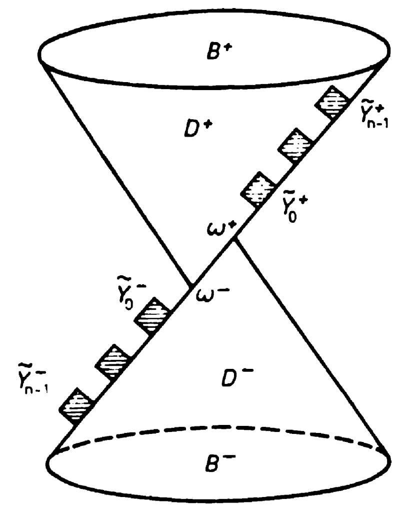

Fig. 2. The scheme of the set Y_n

previously under the definition of Y. The desired set X is defined as the set obtained from this disjoint union by the identifications of the arcs K_{n+1}^+ and I_{n+1}^+ as well as K_{n+1}^- and I_{n+1}^-, $n = 0, 1, 2, \ldots$, where $I_{n+1}^+ \subset \tilde{Y}_n^+$ and $I_{n+1}^- \subset \tilde{Y}_n^-$ are defined as above. It is obvious that X is an acyclic locally connected continuum, which is not an ANR-set.

As mentioned at the beginning of this section, for each θ, $0 < \theta < 2\pi$, there is a homeomorphism h_θ of the set X_0 onto itself, which has no fixed point. With the aid of h_θ we can define a map f_θ of the set X into itself, which has no fixed point. Namely, we define f_θ as the composition $h_\theta \circ r$, where r is a retraction of X onto X_0.

It remains to prove that the sets X and Y are quasi-homeomorphic.

First, let us prove that X is Y-like. Given an $\varepsilon > 0$, we must construct an ε-mapping of X onto Y. For this purpose, we first define a space X_n and a map f_n of X onto X_n for any $n = 1, 2, \ldots$ Given an n, let L_n be an arc with the end-points q_n^+, q_n^- contained in the spiral $S \subset X_0$ such that K_i^+, $K_i^- \subset L_n$ for $i < n$, $K_i^+ \cap L_n = \emptyset = K_i^- \cap L_n$ for $i \geqslant n$, q_n^+, $q_n^- \in I$ and q_n^+ has a positive z-coordinate. (This arc can be found if the arcs K_i^+, K_i^- are suitably placed in S.) Hence $S = L_n \cup S_n^+ \cup S_n^-$, where S_n^+ and S_n^- are the closures of the respective components of $S \setminus L_n$. Recall that $X_0 = \tilde{C}^+ \cup \tilde{C}^-$, where $\tilde{C}^+$ and $\tilde{C}^-$ are (topological) infinite cones without vertices. Divide $\tilde{C}^-$ by the horizontal plane passing through q_n^+ into the union of two closed (in

$\tilde{C}^-$) sets C_n^- and E_n^-, where $C_n^- \cap \tilde{C}^+ = S_n^- \cup L_n$, $E_n^- \cap \tilde{C}^+ = S_n^+$. Then do the same for $\tilde{C}^+$. Hence C_n^- and C_n^+ are closed cylinders, but E_n^- and E_n^+ are infinite cones without vertices. Now, consider two geometric cones F_n^+, F_n^-, whose bases respectively are B^+ and B^- and whose vertices respectively are q_n^- and q_n^+. Then define f_n on the set $X_0 \subset X$ so that $f_n(X_0) = F_n^+ \cup F_n^-$, $f_n(C_n^-) = F_n^-$, $f_n(C_n^+) = F_n^+$, $f_n | B^+ \cup B^- = \mathrm{id}$, $f_n | (C_n^+ \setminus E_n^+ \cup C_n^- \setminus E_n^-)$ is a homeomorphism and each intersection of E_n^+ (E_n^-) with a horizontal plane is mapped by f_n to a point. Next, we extend this map to a map (denoted also by f_n) of X into the Euclidean 3-space E^3 such that $f_n | \bigcup_{i<n} (\tilde{Y}_i^+ \cup \tilde{Y}_i^-)$ is a homeomorphism, $f_n(\tilde{Y}_i^+)$ intersects $F_n^+ \cup F_n^-$ on the set $f_n(\tilde{Y}_i^+ \cap X_0)$ and the same holds for the set $f_n(\tilde{Y}_i^-)$, and moreover $f_n(\tilde{Y}_i^+) = f_n(\tilde{Y}_i^+ \cap X_0)$, $f_n(\tilde{Y}_i^-) = f_n(\tilde{Y}_i^- \cap X_0)$ for $i \geqslant n$. Then define X_n to be equal to $f_n(X)$ (cf. Fig. 3).

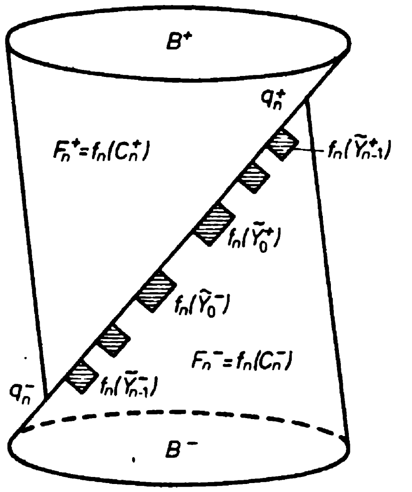

Fig. 3. The scheme of the set X_n

Now, given an $\varepsilon > 0$, fix n so large that

(1) $$f_n \text{ is a } \tfrac{1}{6}\varepsilon\text{-mapping.}$$

To find an ε-mapping of X onto Y, we shall construct one more space X_n' and two maps $g\colon X_n \twoheadrightarrow X_n'$, $h\colon X_n' \twoheadrightarrow Y$ where $\twoheadrightarrow$ denotes the surjection. The desired ε-mapping will be the composition $h \circ g \circ f_n$.

We shall construct the set X_n' by attaching some segments to the set X_n. Observe that $f_n(L_n) = f_n(C_n^+ \cap C_n^-) = F_n^+ \cap F_n^-$ is the segment with the end-points q_n^+ and q_n^- contained in the segment I joining v^+ and v^-. Now, divide the segment I into the union of smaller segments $I_1', I_2', \ldots, I_p'$ (successively ordered on I) such that

(2) $$\operatorname{diam}[f_n^{-1}(I_j') \cap (C_n^+ \cup C_n^-)] < \tfrac{1}{6}\varepsilon \quad \text{for each } j \leqslant p.$$

Next, find a segment M_j going from the center m_j of the segment I'_j and disjoint from X_n except for this point. Then X'_n is defined by the formula

$$X'_n = X_n \cup \bigcup_{j \leqslant p} M_j.$$

To construct the above-mentioned map $g\colon X_n \to\to X'_n$, for any point m_j we find a disk $Q_j \subset F_n^+ \cup F_n^- \subset X_n$ such that $Q_j \cap I = \{m_j\}$, that for any point $x \in Q_j \setminus \{m_j\}$ the set $f_n^{-1}(x)$ consists of one point, and so small that

(3) $$\operatorname{diam} f_n^{-1}(Q_j \setminus \{m_j\}) < \tfrac{1}{6}\varepsilon.$$

Then define the map g so that it maps homeomorphically the set $X_n \setminus \bigcup_{j \leqslant p} Q_j$ onto the set $X'_n \setminus \bigcup_{j \leqslant p} M_j$, $g|I = \mathrm{id}$, $g(\mathring{Q}_j) = \{m_j\}$ and $g(Q_j) = M_j$.

Now, we shall construct the second above-mentioned map $h\colon X'_n \to \to Y$. First, observe that for $n > 0$ the set X_n is (naturally) homeomorphic with a subset of the set $\tilde{Y}_n^+ = Y_{0n}^+ \cup Y_n^+ \subset Y$ being the union of Y_{0n}^+ and of all homothetic images of $\tilde{Y}_i^+$, $\tilde{Y}_i^-$ for $i < n$ contained in Y_n^+ (under the homothetic transformation of Y_n onto Y_n^+). Let h' denote such a linear homeomorphism. The desired map h will be constructed by extending h' onto the set X'_n. For this purpose, consider the segment $h'(I)$ and divide the set $\overline{Y \setminus h'(X_n)}$ into the union of p closed slices by means of $p-1$ horizontal planes passing through the images under h' of the points dividing I into the union $\bigcup_{j \leqslant p} I'_j$. Obviously the slices are locally connected continua. For any point $h'(m_j)$, where $1 \leqslant j \leqslant p$, there is exactly one slice containing this point, which we shall denote by S_j. Since S_j is a locally connected continuum, there is a map h'_j of the segment M_j onto S_j. Modifying the maps h'_j, $j \leqslant p$, if necessary, we can assume that they agree with the map h', i.e. that $h'(m_j) = h'_j(m_j)$ for $j \leqslant p$. Then the maps h' and h'_j, $j \leqslant p$, define a map of X'_n onto Y, which is the required map h.

It remains to verify that the composition $h \circ g \circ f_n$ is an ε-mapping. Consider a point $y \in Y$. If the set $h^{-1}(y)$ consists of more than one point, then y belongs to one slice S_j or two consecutive slices S_j, S_{j+1}. Then $h^{-1}(y)$ is contained in $I'_j \cup M_j$ or in the union $I'_j \cup M_j \cup I'_{j+1} \cup M_{j+1}$ for some j. By the construction of g, in the last case the set $g^{-1}h^{-1}(y)$ is contained in $I'_j \cup Q_j \cup I'_{j+1} \cup Q_{j+1}$. When $h^{-1}(y)$ is not contained in the set $I \cup \bigcup_{j \leqslant p} M_j$, then $g^{-1}h^{-1}(y)$ is a point. Since f_n is a $\frac{1}{6}\varepsilon$-mapping by (1), in this case $\operatorname{diam}(f_n^{-1}g^{-1}h^{-1}(y)) < \frac{1}{6}\varepsilon$. Thus to prove that $\operatorname{diam} f_n^{-1}g^{-1}h^{-1}(y) < \varepsilon$ for each $y \in Y$, it suffices to observe that

(4) $$\operatorname{diam} f_n^{-1}(I'_j \cup Q_j) < \tfrac{1}{2}\varepsilon \quad \text{for any } j \leqslant p.$$

But the set $f_n^{-1}(I'_j \cup Q_j)$ is the union of the following sets: $f_n^{-1}(I'_j) \cap (C_n^- \cup C_n^+)$, $f_n^{-1}(Q_j \setminus \{m_j\})$ and of some non-degenerate sets of the form $f_n^{-1}(x)$, where $x \in I'_j$. Because of (1), (2) and (3), each of these sets has diameter less than $\frac{1}{6}\varepsilon$. By the construction of the disks Q_j, $j \leqslant p$, and of the map f_n, the closure of the second set, as well as any one of the sets of the third kind have a common point with the first set. It follows that the inequality (4) holds, which completes the proof that the map $h \circ g \circ f_n$ is an ε-mapping.

Now, we must show that Y is X-like, i.e. given an $\varepsilon > 0$, we must construct an ε-mapping of Y onto X. The idea of the construction is the same as in the previous case, so we shall only sketch it. Denote by r_n a retraction of Y onto Y_n which maps each of the sets $\tilde{Y}_i^+$, $\tilde{Y}_i^-$, $i \geqslant n$ contained in Y onto its common segment with $Y_0 \subset Y_n$. Fix n so large that r_n is a $\frac{1}{6}\varepsilon$-mapping. Divide the segment $I \subset Y_0 \subset Y_n$ into q segments $I''_1, \ldots, I''_q$ such that diam $I''_j < \frac{1}{6}\varepsilon$ for $j \leqslant q$ and construct a space Y'_n by attaching to Y_n of q segments going from the centers of the segments I''_j, $j \leqslant q$. Then construct a $\frac{1}{6}\varepsilon$-mapping φ of Y_n onto Y'_n in the same way as we previously constructed the map g of X_n onto X'_n. Finally, we construct a map ψ of Y'_n onto X similarly as we previously constructed the map h of X'_n onto Y. Namely, observe that the set Y_n is (naturally) homeomorphic with a subset of the set $\tilde{Y}_n^+$ (considered as a subset of X), because $\tilde{Y}_n^+ = Y_{0n}^+ \cup Y_n^+$, where Y_n^+ is a homothetic image of Y_n. Denote this homeomorphism of Y_n into $\tilde{Y}_n^+ \subset X$ by ψ' and extend ψ' to a map ψ of Y'_n onto X by dividing the rest of X (i.e. of the closure of the set $X \setminus \psi'(Y_n)$) into q closed slices and by proceeding as above. It is clear that the composition $\psi \circ \varphi \circ r_n$ is the desired ε-mapping of Y onto X.

This completes the proof of the following theorem:

THEOREM. *There are two quasi-homeomorphic 2-dimensional compacta X and Y such that Y is an AR-set and X does not have the fixed-point property.*

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