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ON NORMAL AGASSIZ SYSTEMS OF ALGEBRAS

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The concepts of Agassiz system of algebras and Agassiz sum were introduced by Grätzer and Sichler in [3]. In this paper we intend to discuss a modification of these concepts, which seems to be more advantageous.

Our notation and nomenclature are basically those of Grätzer [1] but we prefer to have the symbol $=$ reserved exclusively for the "real" equality, and thus, by identities of a given type τ we mean expressions of the form $\boldsymbol{p} \equiv \boldsymbol{q}$, where $\boldsymbol{p}$ and $\boldsymbol{q}$ are polynomial symbols built up in the usual way (see [1]) by some variables from the list $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots$ and some operation symbols of type τ. The set of such polynomial symbols will be denoted by $\boldsymbol{P}(\tau)$ and they will be referred to as polynomial symbols of type τ.

A mapping $N\colon \boldsymbol{P}(\tau) \to \boldsymbol{P}(\varrho)$ will be called a *naming functor* (cf. [3]) if for every $\boldsymbol{p} \in \boldsymbol{P}(\tau)$ the variables of $\boldsymbol{p}$ and $N(\boldsymbol{p})$ are the same. Let us note that to assure the existence of a naming functor from $\boldsymbol{P}(\tau)$ into $\boldsymbol{P}(\varrho)$ it is necessary and sufficient to require that the range of type ϱ contains 0 whenever the range of τ does and the range of ϱ contains a number greater than 1 whenever the range of τ does.

Given a naming functor $N\colon \boldsymbol{P}(\tau) \to \boldsymbol{P}(\varrho)$, we say that an algebra $\mathfrak{B}$ of type ϱ belongs to the *structurality class* of N ($\mathfrak{B} \in \mathrm{SC}(N)$) if for every n-ary ($n \geqslant 1$) polynomial symbol $\boldsymbol{p} \in \boldsymbol{P}(\tau)$ and for every $\boldsymbol{q}_1, \ldots, \boldsymbol{q}_n \in \boldsymbol{P}(\tau)$ the following conditions hold:

(i) $N(\boldsymbol{p}(\boldsymbol{q}_1, \ldots, \boldsymbol{q}_n)) \equiv N(\boldsymbol{p})(N(\boldsymbol{q}_1), \ldots, N(\boldsymbol{q}_n)) \in \mathrm{Id}(\mathfrak{B})$;

(ii) $N(\boldsymbol{p}) \equiv \boldsymbol{p} \in \mathrm{Id}(\mathfrak{B})$ whenever $\boldsymbol{p}$ is a variable.

Observe that $\mathrm{SC}(N)$ always is an equational class of algebras.

An identity $\boldsymbol{p} \equiv \boldsymbol{q}$ of type τ is called *N-regular* in a class of algebras I of type ϱ (cf. [3]) if $N(\boldsymbol{p}) \equiv N(\boldsymbol{q}) \in \mathrm{Id}(I)$. Let the symbol $\mathrm{Id}_N(I)$ denote the set of all N-regular identities in I. It is easy to see that $\mathrm{Id}_N(I)$ is a closed set of identities whenever $I \subseteq \mathrm{SC}(N)$.

An identity $\boldsymbol{p} \equiv \boldsymbol{q}$ will be called *symmetric* if $\boldsymbol{p} = \boldsymbol{q}$ or if none of $\boldsymbol{p}$ and $\boldsymbol{q}$ is a variable. If Σ is a set of identities, then by $\mathrm{Sm}(\Sigma)$ we denote the set of all symmetric identities from Σ. Observe that $\mathrm{Sm}(\Sigma)$ is a closed set of identities whenever Σ is.

An identity will be called *inconsistent* if it holds only in degenerate (one-element) algebras. It is easy to see that an asymmetric identity having no variable occurring jointly on both its sides must be inconsistent.

Let us define a *normal Agassiz system* of algebras as a quadruplet

$$\mathcal{S} = \big(\mathfrak{B}, (\mathfrak{A}_b \mid b \in B), (h_{bc} \mid \langle b, c\rangle \in R), N\big)$$

such that

(i) $\mathfrak{B}$ is an algebra of type ϱ (indexing algebra of $\mathcal{S}$);

(ii) $(\mathfrak{A}_b \mid b \in B)$ is a family of algebras of type τ (indexed family of $\mathcal{S}$) such that $A_b \cap A_c = \emptyset$ whenever $b, c \in B$, $b \neq c$;

(iii) $N\colon \boldsymbol{P}(\tau) \to \boldsymbol{P}(\varrho)$ is a naming functor (naming functor of $\mathcal{S}$) such that $\mathfrak{B} \in \mathrm{SC}(N)$;

(iv) $R \subseteq B \times B$ is a transitive relation such that for every n-ary $(n \geqslant 1)$ operation symbol f of type τ if

$$b_1, \ldots, b_n \in B \quad \text{and} \quad b = \big(N(f(x_1, \ldots, x_n))\big)_{\mathfrak{B}}(b_1, \ldots, b_n),$$

then $\langle b_i, b\rangle \in R$, $i = 1, \ldots, n$;

(v) $(h_{bc} \mid \langle b, c\rangle \in R)$ is a family of homomorphisms (h_{bc} is a homomorphism of $\mathfrak{A}_b$ into $\mathfrak{A}_c$) such that $h_{cd} \circ h_{bc} = h_{bd}$ whenever $\langle b, c\rangle, \langle c, d\rangle \in R$.

The *sum* of the normal Agassiz system $\mathcal{S}$ is an algebra $\mathfrak{A}$ of type τ with the base set $A = \bigcup (A_b \mid b \in B)$ and the basic operations defined as follows:

(i) if f is an n-ary $(n \geqslant 1)$ operation symbol of type τ, $a_1, \ldots, a_n \in A$, $a_i \in A_{b_i}$, $i = 1, \ldots, n$, and

$$b = \big(N(f(x_1, \ldots, x_n))\big)_{\mathfrak{B}}(b_1, \ldots, b_n),$$

then

$$(f)_{\mathfrak{A}}(a_1, \ldots, a_n) = (f)_{\mathfrak{A}_b}\big(h_{b_1 b}(a_1), \ldots, h_{b_n b}(a_n)\big);$$

(ii) if f is a nullary operation symbol of type τ, then $(f)_{\mathfrak{A}} = (f)_{\mathfrak{A}_c}$, where $c = \big(N(f)\big)_{\mathfrak{B}}$.

From now on it will always be assumed tacitly that K and I are non-empty classes of algebras of types τ and ϱ, respectively, and that $N\colon \boldsymbol{P}(\tau) \to \boldsymbol{P}(\varrho)$ is a naming functor such that $I \subseteq \mathrm{SC}(N)$. We will use the notation (I, K, N) for the class of all normal Agassiz systems whose naming functor is N, indexing algebras are of class I, and indexed families consist of isomorphic copies of algebras of K. The symbol $\lim(I, K, N)$ will be used to denote the class of all isomorphic copies of sums of normal Agassiz systems of (I, K, N).

We are not sure that our understanding of the definition of the Agassiz system and Agassiz sum as stated in [3] is in accordance with the authors' intention but, nevertheless, we have an evidence that the concepts of [3] do not coincide with ours, since we have only the following weakened version of the Theorem of [3]:

PROPOSITION 1. $\mathrm{Sm}(\mathrm{Id}_N(I)\cap\mathrm{Id}(K)) \subseteq \mathrm{Id}(\lim(I,K,N)) \subseteq \mathrm{Id}_N(I)\cap \cap\mathrm{Id}(K)$.

The reader will have no difficulty in proving the proposition with the help of the following

LEMMA. *Let $\mathfrak{A}$ be the sum of a normal Agassiz system $\mathcal{S}$. Let $\boldsymbol{p}$ be an n-ary ($n \geqslant 1$) polynomial symbol of type of $\mathfrak{A}$. Let $a_1, \ldots, a_n \in A$, $a_i \in A_{b_i}$, $i = 1, \ldots, n$, and $(\boldsymbol{p})_{\mathfrak{A}}(a_1, \ldots, a_n) \in A_b$. Then the following conditions are satisfied:*

(i) $b = (N(\boldsymbol{p}))_{\mathfrak{B}}(b_1, \ldots, b_n)$;

(ii) *if $\langle b, c\rangle \in R$ and x_i is a variable of $\boldsymbol{p}$, then $\langle b_i, c\rangle \in R$;*

(iii) *if $\boldsymbol{p}$ is not a variable and x_i is a variable of $\boldsymbol{p}$, then $\langle b_i, b\rangle \in R$;*

(iv) *if $\langle b, c\rangle, \langle b_1, c\rangle, \ldots, \langle b_n, c\rangle \in R$, then*

$$h_{bc}\big((\boldsymbol{p})_{\mathfrak{A}}(a_1, \ldots, a_n)\big) = (\boldsymbol{p})_{\mathfrak{A}_c}\big(h_{b_1c}(a_1), \ldots, h_{b_nc}(a_n)\big);$$

(v) *if $\boldsymbol{p}$ is not a variable, then*

$$(\boldsymbol{p})_{\mathfrak{A}}(a_1, \ldots, a_n) = (\boldsymbol{p})_{\mathfrak{A}_b}(a_1^*, \ldots, a_n^*), \qquad \text{where } a_i^* = h_{b_ib}(a_i),$$

whenever x_i is a variable of $\boldsymbol{p}$ and a_i^* is an arbitrary element of A_b otherwise.

Proof. Conditions (i), (ii), and (iv) can be verified by a routine induction argument on the rank of $\boldsymbol{p}$. Condition (iii) is an easy consequence of (ii). To prove (v) observe first that it holds trivially if $\boldsymbol{p}$ is a nullary operation symbol. Now, let us suppose that $\boldsymbol{p} = \boldsymbol{f}(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_m)$, where $\boldsymbol{f}$ is an m-ary ($m \geqslant 1$) operation symbol and $\boldsymbol{p}_1, \ldots, \boldsymbol{p}_m$ are n-ary polynomial symbols. Without loss of generality we can assume that all the variables of $\boldsymbol{p}$ are $x_1, \ldots, x_k$ for some $k \leqslant n$. Let us compute

$$\begin{aligned}(\boldsymbol{p})_{\mathfrak{A}}(a_1, \ldots, a_n) &= (\boldsymbol{p})_{\mathfrak{A}}(a_1, \ldots, a_k)\\ &= (\boldsymbol{f})_{\mathfrak{A}}\big((\boldsymbol{p}_1)_{\mathfrak{A}}(a_1, \ldots, a_k), \ldots, (\boldsymbol{p}_m)_{\mathfrak{A}}(a_1, \ldots, a_k)\big)\\ &= (\boldsymbol{f})_{\mathfrak{A}_c}\Big(h_{c_1c}\big((\boldsymbol{p}_1)_{\mathfrak{A}}(a_1, \ldots, a_k)\big), \ldots, h_{c_mc}\big((\boldsymbol{p}_m)_{\mathfrak{A}}(a_1, \ldots, a_k)\big)\Big),\end{aligned}$$

where

$$(\boldsymbol{p}_j)_{\mathfrak{A}}(a_1, \ldots, a_k) \in A_{c_j}, \qquad j = 1, \ldots, m,$$

and

$$c = \big(N\big(\boldsymbol{f}(x_1, \ldots, x_m)\big)\big)_{\mathfrak{B}}(c_1, \ldots, c_m).$$

Thus $(\boldsymbol{p})_{\mathfrak{A}}(a_1, \ldots, a_n) \in A_c$, which means that $c = b$ and, therefore, $\langle c_j, b\rangle \in R$ for every $j = 1, \ldots, m$. Since by (iii) we know that $\langle b_i, b\rangle \in R$

for every $i = 1, \ldots, k$, applying (iv) we get

$$\begin{aligned}(p)_{\mathfrak{A}}(a_1, \ldots, a_n) &= (f)_{\mathfrak{A}_b}\big(h_{c_1 b}((p_1)_{\mathfrak{A}}(a_1, \ldots, a_k)), \ldots, h_{c_m b}((p_m)_{\mathfrak{A}}(a_1, \ldots, a_k))\big)\\ &= (f)_{\mathfrak{A}_b}\big((p_1)_{\mathfrak{A}_b}(h_{b_1 b}(a_1), \ldots, h_{b_k b}(a_k)), \ldots, (p_m)_{\mathfrak{A}_b}(h_{b_1 b}(a_1), \ldots, h_{b_k b}(a_k))\big)\\ &= \big(f(p_1, \ldots, p_m)\big)_{\mathfrak{A}_b}(h_{b_1 b}(a_1), \ldots, h_{b_k b}(a_k))\\ &= (p)_{\mathfrak{A}_b}(a_1^*, \ldots, a_k^*) = (p)_{\mathfrak{A}_b}(a_1^*, \ldots, a_n^*)\end{aligned}$$

as it was claimed. This completes the proof.

Following Grätzer [2], by a *retraction* of an algebra $\mathfrak{A}$ we mean an endomorphism $g\colon \mathfrak{A} \to \mathfrak{A}$ such that $g \circ g = g$. A retraction g is *non-trivial* if $g(a) \neq a$ for some $a \in A$. In order to get a characterization of identities holding in $\lim(I, K, N)$ let us note the following sharpened version of Proposition 1:

PROPOSITION 2. (i) *If algebras of K have only trivial retractions, then*

$$\mathrm{Id}\big(\lim(I, K, N)\big) = \mathrm{Id}_N(I) \cap \mathrm{Id}(K).$$

(ii) *If an algebra of K has a non-trivial retraction, then*

$$\mathrm{Id}\big(\lim(I, K, N)\big) = \mathrm{Sm}\big(\mathrm{Id}_N(I) \cap \mathrm{Id}(K)\big).$$

Proof. To prove (i) observe first that it holds trivially if every algebra of I is degenerate. Suppose that I contains a non-degenerate algebra and

$$\mathrm{Id}\big(\lim(I, K, N)\big) \neq \mathrm{Id}_N(I) \cap \mathrm{Id}(K).$$

Then, from Proposition 1 we know that some identity of the form $p \equiv x_k$, where p is not a variable, belongs to $\mathrm{Id}_N(I) \cap \mathrm{Id}(K)$, but does not hold in $\lim(I, K, N)$. Obviously, the variable x_k must occur in p since, in the opposite case, $N(p) \equiv x_k$ is inconsistent and holds in I. Without loss of generality we can assume that all the variables of p are $x_1, \ldots, x_n$ for some $n \geqslant k$. Let the algebra $\mathfrak{A} \in \lim(I, K, N)$ be such that $p \equiv x_k \notin \mathrm{Id}(\mathfrak{A})$; then $\mathfrak{A}$ is a sum of a normal Agassiz system $\mathscr{S}$ whose indexing algebra belongs to I and indexed family consists of isomorphic copies of algebras of K. Let $a_1, \ldots, a_n \in A$, where $a_i \in A_{b_i}$, be such that

$$(p)_{\mathfrak{A}}(a_1, \ldots, a_n) \neq (x_k)_{\mathfrak{A}}(a_1, \ldots, a_n).$$

Suppose that $(p)_{\mathfrak{A}}(a_1, \ldots, a_n) \in A_b$. Then, by Lemma (iii), $\langle b_i, b\rangle \in R$ for every $i = 1, \ldots, n$, and, by Lemma (v),

$$\begin{aligned}(p)_{\mathfrak{A}}(a_1, \ldots, a_n) &= (p)_{\mathfrak{A}_b}\big(h_{b_1 b}(a_1), \ldots, h_{b_n b}(a_n)\big)\\ &= (x_k)_{\mathfrak{A}_b}\big(h_{b_1 b}(a_1), \ldots, h_{b_n b}(a_n)\big) = h_{b_k b}(a_k).\end{aligned}$$

Next, by Lemma (i),

$$b = \big(N(p)\big)_{\mathfrak{B}}(b_1, \ldots, b_n) = \big(N(x_k)\big)_{\mathfrak{B}}(b_1, \ldots, b_n) = (x_k)_{\mathfrak{B}}(b_1, \ldots, b_n) = b_k$$

which gives

$$h_{bb}(a_k) = h_{b_k b}(a_k) = (p)_{\mathfrak{A}}(a_1, \ldots, a_n) \neq (x_k)_{\mathfrak{A}}(a_1, \ldots, a_n) = a_k$$

proving that h_{bb} is a non-trivial retraction of $\mathfrak{U}_b$.

To prove (ii) pick $\mathfrak{B} \in I$ and a family $(\mathfrak{U}_b \mid b \in B)$ composed of isomorphic copies of an algebra $\mathfrak{A}_0 \in K$ having a non-trivial retraction g. For every $b \in B$ let f_b be an isomorphism of $\mathfrak{U}_b$ onto $\mathfrak{A}_0$. Define a family of homomorphisms $(h_{bc} \mid b, c \in B)$ by putting $h_{bc} = f_c^{-1} \circ g \circ f_b$. Then

$$\mathscr{S} = \big(\mathfrak{B}, (\mathfrak{A}_b \mid b \in B), (h_{bc} \mid \langle b, c\rangle \in B), N\big)$$

is a normal Agassiz system of (I, K, N). We will show that no asymmetric identity of $\mathrm{Id}_N(I) \cap \mathrm{Id}(K)$ holds in the algebra $\mathfrak{A}$ being the sum of the system $\mathscr{S}$. Indeed, suppose that $p \equiv x_k$ is such an identity. Then the variable x_k must occur in p since, in the opposite case, $p \equiv x_k$ is inconsistent and fails to hold in $\mathfrak{A}_0$ being non-degenerate. As before, we can assume that all the variables of p are $x_1, \ldots, x_n$ for some $n \geqslant k$. Let $a_0 \in A_0$ be such that $g(a_0) \neq a_0$. Pick $b_1, \ldots, b_n \in B$ and $a_i \in A_{b_i}$, $i = 1, \ldots, n$, such that $a_k = f_{b_k}^{-1}(a_0)$. Applying Lemma (i), (v), we get

$$\begin{aligned}(p)_{\mathfrak{A}}(a_1, \ldots, a_n) &= (p)_{\mathfrak{A}_b}\big(h_{b_1 b}(a_1), \ldots, h_{b_n b}(a_n)\big)\\ &= (x_k)_{\mathfrak{A}_b}\big(h_{b_1 b}(a_1), \ldots, h_{b_n b}(a_n)\big) = h_{b_k b}(a_k),\end{aligned}$$

where

$$b = \big(N(p)\big)_{\mathfrak{B}}(b_1, \ldots, b_n) = \big(N(x_k)\big)_{\mathfrak{B}}(b_1, \ldots, b_n) = (x_k)_{\mathfrak{B}}(b_1, \ldots, b_n) = b_k.$$

Thus

$$\begin{aligned}(p)_{\mathfrak{A}}(a_1, \ldots, a_n) &= h_{b_k b_k}(a_k) = f_{b_k}^{-1} \circ g \circ f_{b_k} \circ f_{b_k}^{-1}(a_0) = f_{b_k}^{-1}\big(g(a_0)\big) \neq f_{b_k}^{-1}(a_0)\\ &= a_k = (x_k)_{\mathfrak{A}}(a_1, \ldots, a_n)\end{aligned}$$

proving that $p \equiv x_k$ does not hold in $\mathfrak{A}$, which completes the proof.

Let us agree to use the symbol T for a trivial naming functor such that $T(p) = p$ for every polynomial symbol p of a considered type. By Propositions 1 and 2 (ii) we get the following

COROLLARY. *An equational class K containing a non-degenerate algebra can be defined by means of symmetric identities if and only if* $\lim(K, K, T) \subseteq K$.

Proof. The necessity follows directly from Proposition 1. To prove the sufficiency observe that the free algebra $\mathfrak{F}_K(2)$ must be non-degenerate if there is a non-degenerate algebra in K. Let a_1 and a_2 be free generators of $\mathfrak{F}_K(2)$ and let $g\colon \mathfrak{F}_K(2) \to \mathfrak{F}_K(2)$ be an endomorphism such that $a_1 = g(a_1) = g(a_2)$. Then g is a non-trivial retraction of $\mathfrak{F}_K(2)$, and thus if $\lim(K, K, T) \subseteq K$, then

$$\mathrm{Id}(K) \subseteq \mathrm{Id}\big(\lim(K, K, T)\big) = \mathrm{Sm}\big(\mathrm{Id}(K)\big)$$

by Proposition 2 (ii). This completes the proof.

To get a characterization of $\mathrm{Id}_N(I) \cap \mathrm{Id}(K)$ in terms of normal Agassiz systems we need a refinement of the class (I, K, N). Let us say that a normal Agassiz system $\mathcal{S}$ is *fine* if it satisfies the following condition:

(F) for every $b \in B$ there exists $c \in B$ such that $\langle b, c\rangle \in R$ and h_{bc} is an embedding.

Denoting by $(I, K, N)_F$ the class of all fine Agassiz systems from (I, K, N) we can state the following

PROPOSITION 3. $\mathrm{Id}\big(\lim(I, K, N)_F\big) = \mathrm{Id}_N(I) \cap \mathrm{Id}(K)$.

Proof. The inclusion $\mathrm{Id}\big(\lim(I, K, N)_F\big) \subseteq \mathrm{Id}_N(I) \cap \mathrm{Id}(K)$ is almost obvious. To prove the converse let us suppose that $\mathfrak{A}$ is the sum of a fine Agassiz system $\mathcal{S}$ from the class $(I, K, N)_F$. Suppose that the identity

$$p \equiv q \in \mathrm{Id}_N(I) \cap \mathrm{Id}(K)$$

does not hold in $\mathfrak{A}$. By Proposition 1, the identity $p \equiv q$ must be asymmetric which yields that either p or q contains a variable. Without loss of generality we can assume that $x_1, \ldots, x_n$ are all the variables of p and q. Pick $a_1, \ldots, a_n \in A$, $a_i \in A_{b_i}$, $i = 1, \ldots, n$, such that

$$(p)_{\mathfrak{A}}(a_1, \ldots, a_n) \neq (q)_{\mathfrak{A}}(a_1, \ldots, a_n).$$

Suppose that $(p)_{\mathfrak{A}}(a_1, \ldots, a_n) \in A_b$. Then, by Lemma (i),

$$b = \big(N(p)\big)_{\mathfrak{B}}(b_1, \ldots, b_n) = \big(N(q)\big)_{\mathfrak{B}}(b_1, \ldots, b_n)$$

which implies that also $(q)_{\mathfrak{A}}(a_1, \ldots, a_n) \in A_b$. Now, applying (F) we can pick $c \in B$ such that $\langle b, c\rangle \in R$ and h_{bc} is an embedding of $\mathfrak{A}_b$ into $\mathfrak{A}_c$. From Lemma (ii) it follows that $\langle b_i, c\rangle \in R$ for every $i = 1, \ldots, n$, and, finally, by Lemma (iv) we have

$$\begin{aligned}&(p)_{\mathfrak{A}_c}\big(h_{b_1c}(a_1), \ldots, h_{b_nc}(a_n)\big)\\ &= h_{bc}\big((p)_{\mathfrak{A}}(a_1, \ldots, a_n)\big) \neq h_{bc}\big((q)_{\mathfrak{A}}(a_1, \ldots, a_n)\big) = (q)_{\mathfrak{A}_c}\big(h_{b_1c}(a_1), \ldots, h_{b_nc}(a_n)\big)\end{aligned}$$

proving that $p \equiv q$ fails in $\mathfrak{A}_c$, a contradiction.

It is easy to see that fine Agassiz systems of algebras can be treated as a generalization of direct systems considered by Płonka in [4], and Proposition 3 as a generalization of Theorem 1 of [4] (cf. [3]).

Investigating classes of algebras of kind $\lim(I, K, N)$ from the algebraic point of view we can confine ourselves to the cases where K and I are of the same type and the naming functor N is trivial. To be more exact we give the following

PROPOSITION 4. *There exists a class of algebras I_N of the same type as K such that*

$$\lim(I, K, N) = \lim(I_N, K, T) \quad \textit{and} \quad \mathrm{Id}_N(I) = \mathrm{Id}(I_N).$$

Proof. For every $\mathfrak{B} \in I$ we define an algebra $\mathfrak{B}_N$ of type τ with the base set B and the basic operations such that: if f is an n-ary ($n \geqslant 1$) operation symbol of type τ and $b_1, \ldots, b_n \in B$, then

$$(f)_{\mathfrak{B}_N}(b_1, \ldots, b_n) = \big(N(f(x_1, \ldots, x_n))\big)_{\mathfrak{B}}(b_1, \ldots, b_n);$$

if f is a nullary operation symbol of type τ, then

$$(f)_{\mathfrak{B}_N} = \big(N(f)\big)_{\mathfrak{B}}.$$

It is easy to check that the class $I_N = \{\mathfrak{B}_N \mid \mathfrak{B} \in I\}$ has all the required properties.

Proposition 4 shows, in fact, that naming functors can be eliminated and suggests the following definition.

A triplet

$$\mathscr{S} = \big(\mathfrak{B}, (\mathfrak{A}_b \mid b \in B), (h_{bc} \mid \langle b, c\rangle \in R)\big)$$

is a *normal system* of algebras if

$$\mathscr{S}' = \big(\mathfrak{B}, (\mathfrak{A}_b \mid b \in B), (h_{bc} \mid \langle b, c\rangle \in R), T\big)$$

is a normal Agassiz system of algebras. The *sum* of the normal system $\mathscr{S}$ is defined as that of $\mathscr{S}'$. For I and K being non-empty classes of algebras of the same type we write $\lim(I, K)$ and $\lim(I, K)_{\mathrm{F}}$ instead of $\lim(I, K, T)$ and $\lim(I, K, T)_{\mathrm{F}}$, respectively. An immediate consequence of the facts stated previously is the following

PROPOSITION 5. (i) *If algebras of K have only trivial retractions, then*

$$\mathrm{Id}\big(\lim(I, K)\big) = \mathrm{Id}(I \cup K).$$

(ii) *If an algebra of K has a non-trivial retraction, then*

$$\mathrm{Id}\big(\lim(I, K)\big) = \mathrm{Sm}\big(\mathrm{Id}(I \cup K)\big).$$

(iii) $\mathrm{Id}\big(\lim(I, K)_{\mathrm{F}}\big) = \mathrm{Id}(I \cup K)$.

(iv) *An identity is preserved under the formation of sums of normal systems of algebras if and only if it is symmetric or inconsistent.*

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