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A KRONECKER THEOREM IN FUNCTIONAL ANALYSIS

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0. Let (x_n, y_n) be a sequence of pairs of elements of R^2 and (P) this property of a subset E of R^2:

(P) Each pair (f_1, f_2) of continuous functions on E to the torus T can be approximated uniformly on E by pairs of characters $\big(\exp 2\pi i(x\cdot x_n), \exp 2\pi i(x\cdot y_n)\big)$.

When E satisfies (P) and has a point of accumulation, there is a pair (f_1, f_2) such that no combination $f_1^p f_2^q$ with integral $(p, q) \neq (0, 0)$ fulfills a Lipschitz condition on E; thus $f_1^p f_2^q$ is not an exponential and there is a subsequence along which

$$(1) \qquad \|px_n + qy_n\| \to +\infty \quad \text{for each integral } (p, q) \neq (0, 0).$$

THEOREM. *Suppose that g is a positive increasing function on $(0, 1)$ and $\lim u^{-2} g(u) = +\infty$ at $0+$, while (x_n, y_n) is a sequence in $R^2 \times R^2$ subject to* (1). *Then there is a Cantor set E in R^2, of infinite Hausdorff g-measure, having property* (P).

1. First the sequence (x_n, y_n) must be normalized in several respects; for this we use the polar representation of vectors in R^2:

$$x_n = X_n(\cos u_n, \sin u_n), \qquad y_n = Y_n(\sin v_n, \cos v_n),$$

wherein $X_n > 0$, $Y_n > 0$, $0 \leqslant u_n < 2\pi$, $0 \leqslant v_n < 2\pi$. Passing to a subsequence we obtain one of four different cases:

(i) $X_n = o(Y_n)$ or $Y_n = o(X_n)$,

(ii) $X_n \leqslant AY_n \leqslant A^2 X_n$ and further

(iia) $|\sin(u_n - v_n)| \geqslant \delta > 0$,

(iib) $\|x_n - \theta y_n\| = o(Y_n)$ for an irrational θ,

(iic) $\|x_n - \theta y_n\| = o(Y_n)$ for a rational $\theta \neq 0$.

To accomplish a further normalization, we rotate the coordinate axes by an angle φ, to be chosen in cases (i), (iia), and (iib). In cases (i) and (iib) we suppose that u_n and v_n have limits u_0 and v_0, and then choose φ

so that neither $\cos(u_0-\varphi)$ nor $\cos(v_0-\varphi)$ vanishes. In case (iib) we suppose also that $X_n Y_n^{-1}$ converges to a number R in $[A^{-1}, A]$ and that $R\cos(u_0-\varphi)$ is incommensurable with $\cos(v_0-\varphi)$. All, except a denumerable set of numbers φ, have this property, because $\sin(u_0-v_0) \neq 0$. From this stage we suppose the axes chosen so that the relations specified are valid when $\varphi = 0$.

Case (iic), when $r\theta = s$, is easily reduced to case (i). For the sequence (x_n', y_n') defined by the formulae $x_n' = x_n - \theta y_n$ and $y_n' = r^{-1}y_n$ fulfills (1) and (i). When E has property (P) relative to (x_n', y_n'), the inverse relations $x_n = x_n' + \theta y_n$ and $y_n = ry_n'$ show that the original sequence is sufficient to approximate all pairs (f_1, f_2^r) of continuous functions. But E is a Cantor set and the continuous functions on E to T form a divisible group, so that (iic) is derived from (i).

2. To each function g mentioned in the theorem there exists a Cantor set F in $(-\infty, \infty)$ and an infinite sequence N of positive integers, so that $F\times F$ has infinite g-measure and a decomposition property for each n in N:

(2) $F\times F$ is a union $\bigcup H_j$ of sets H_j such that $\operatorname{diam} H_j \leqslant \varepsilon_n X_n^{-1}$, $d(H_j, H_k) \geqslant \varepsilon_n^{-1} X_n^{-1}$ for a sequence $\varepsilon_n \to 0$. A similar decomposition is valid relative to Y_n.

Linear sets F are found among the symmetric sets F_M, M being a set of natural numbers and F_M the set of all sums $\sum_M \pm 2^{-m}$. In fact, let

$$\Gamma(n) \equiv \sum_{m\leqslant n} 1,$$

the counting-function of M. If

$$2\Gamma(n)\log 2 + \log g(2^{-n}) \to +\infty,$$

then $F_M\times F_M$ has infinite g-measure. To obtain the decomposition of $F\times F$ we introduce gaps in M near some of the numbers $\log X_n/\log 2$ and $\log Y_n/\log 2$. (More details appear in [1] and [2].) We can now assume that $N = \{1, 2, 3, \ldots\}$.

Let Q be a closed square containing $F\times F$ and let $C^1(Q)$ be the Banach space of real continuously differentiable functions defined over Q, with norm

$$\|f\| = \sup|f| + \sup\|\operatorname{grad} f\|.$$

Let ψ be a fixed member of $C^1(Q)$; we shall show that for all functions ξ in $C^1(Q)$, except a set of the first category (for quasi-all ξ)

$$\Phi = (\xi, \psi) \text{ transforms } F\times F \text{ onto a set with property (P).}$$

Beginning, in particular, with $\psi(u, v) \equiv v$, we thus have a function $\Phi(\xi, \psi)$ with that property, effecting a diffeomorphism of Q into R^2. Then $\Phi(F \times F)$ has infinite g-measure and our theorem will be proved.

The special property of Φ, namely the requirement that its second co-ordinate be specified arbitrarily in $C^1(Q)$, is not necessary for the theorem stated. However, this property shows clearly that a problem of approximation on a fairly large planar set — determined when $g(u) = O(u^{2-\varepsilon})$ for every $\varepsilon > 0$ — can be resolved by displacements along a single direction.

To prove the statement concerning quasi-all elements in $C^1(Q)$ we construct for each ξ, ψ in $C^1(Q)$, each pair of functions f_1, f_2 into T, and each $\varepsilon > 0$, a function ξ_1 in $C^1(Q)$ so that $\|\xi - \xi_1\| < \varepsilon$ in $C^1(Q)$ while

$$|\exp 2\pi i(\Phi \cdot x_n) - f_1| < \varepsilon, \qquad |\exp 2\pi i(\Phi \cdot y_n) - f_2| < \varepsilon$$

for a certain (x_n, y_n). To simplify this, we set $\Phi_0 = (\xi, \psi)$, $\xi_2 = \xi_1 - \xi$, and use the polar form for x_n and y_n. The inequalities become

$$(3) \qquad |\exp 2\pi i \xi_2 X_n \cos u_n - f_1 \exp 2\pi i(\Phi_0 \cdot x_n)| < \varepsilon,$$

$$(4) \qquad |\exp 2\pi i \xi_2 Y_n \cos v_n - f_2 \exp 2\pi i(\Phi_0 \cdot y_n)| < \varepsilon.$$

Now $F \times F = \bigcup H_j$, where the sets H_j have the properties defined in (2). Let z_j be chosen aritrarily in H_j and let $\xi_3(z_j)$ be chosen so that inequality (3) becomes an equality at z_j (with ξ_3 in place of ξ_2). This can be done with

$$|\xi_3| \leqslant X_n^{-1} |\cos u_n|^{-1} = O(X_n^{-1}),$$

because the axes are so chosen that $\lim \cos u_n = \cos u_0 \neq 0$. As in [2] we define $\xi_3 = \xi_3(z_j)$ on H_j and extend ξ_3 to all of Q so that the extension satisfies the inequalities $|\xi_3| = O(X_n^{-1})$ and $\|\operatorname{grad} \xi_3\| = O(\varepsilon_n)$. Also, equality at the points z_j, and $\operatorname{diam} H_j \leqslant \varepsilon_n X_n^{-1}$, yield inequality (3) on $F \times F$. Of course, the fact that ξ and ψ belong to $C^1(Q)$ enters here in the argument.

In case (i), $X_n = o(Y_n)$, we choose ξ_4 in exactly the same way, with $|\xi_4| = O(Y_n^{-1})$, to obtain inequality (4), using $\xi_2 = \xi_3 + \xi_4$. However, inequality (3) is changed by at most $X_n \sup |\xi_4| = O(X_n Y_n^{-1}) = o(1)$, and so $\xi_2 = \xi_3 + \xi_4$ is effective for (3) and (4).

In cases (iia) and (iib) we form ξ_3 exactly as before, but then proceed differently. The polar representation of x_n and y_n and, in case (iib), the relation $\|x_n - \theta y_n\| = o(Y_n)$ yield

$$Y_n \cos v_n = \theta_n X_n \cos u_n, \qquad \lim \theta_n = \theta.$$

In case (iia), $\theta_n = (R_n \cos u_n)^{-1} \cos v_n$ and again $\lim \theta_n$ is irrational and may be named θ.

There is a constant B, depending *only* on θ and ε such that to each real t there is an integer k in $[0, B]$ so that $|k\theta - t| < \frac{1}{2}\varepsilon$ (modulo 1). We attempt to construct ξ_2 in the form $\xi_3 + k(z)(X_n \cos u_n)^{-1}$, where k is an integer function into $[0, B]$, constant on the sets H_j. Of course, this does not affect inequality (3) in any way, since $k(z)$ is integral. Also

$$\xi_2 Y_n \cos v_n = Y_n \xi_3 \cos v_n + \theta_n k(z).$$

Thus we can choose $k(z)$ so that inequality (4) holds at the points z_j, with an error $< \frac{1}{2}\varepsilon$. Now $|\xi_2| \leqslant |\xi_3| + B(X_n \cos u_n)^{-1}$, so the same analysis as before shows that inequality (4) is attained on $F \times F$, and that ξ_2 can be extended to a function of class $C^1(Q)$ of small norm. This completes the proof.

REFERENCES

[1] R. Kaufman, *A functional method for linear sets*, Israel Journal of Mathematics 5 (1967), p. 185-187.

[2] — *Metric properties of some planar sets*, Colloquium Mathematicum 23 (1971), p. 117-120.

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