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A REPRESENTATION OF RELATIVELY COMPLEMENTED DISTRIBUTIVE LATTICES

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It is known [1] that every distributive lattice can be imbedded in a Boolean lattice. It is shown in this paper that, for every relatively complemented distributive lattice L, there exists a Boolean lattice B such that L is equal to the intersection of a prime ideal and an ultrafilter of B (Theorem 1.4 (b)). We also show that the free relatively complemented extension of any distributive lattice exists. Using this result, we give a characterization of free relatively complemented distributive lattices (Theorem 2.5).

0. Background. The following categories will be considered in this paper:

the category $\mathfrak{D}$ whose objects are distributive lattices and whose morphisms are lattice homomorphisms;

the category $\mathfrak{R}$ whose objects are relatively complemented distributive lattices and whose morphisms are lattice homomorphisms;

the category $\mathfrak{B}$ whose objects are Boolean lattices and whose morphisms are lattice homomorphisms.

Let $L \in \mathfrak{D}$ and let $a, b \in L$. Then $a+b$ will denote the join of a and b, ab the meet of a and b, a^* the set of all proper prime ideals of L which do not contain a, and $L^* = \{a^* : a \in L\}$. The lattice L itself will be considered as a prime ideal and an ultrafilter of L.

It is well known (see [1]) that the ring of sets L^* is lattice-isomorphic to L and that if $\hat{B}(L)$ is the field of sets generated by L^*, then $\langle \hat{B}(L), i(L)\rangle$ is the free Boolean lattice extension of L, where $i(L) : L \to \hat{B}(L)$ is the natural imbedding of L in $\hat{B}(L)$. That is, if $B_1 \in \mathfrak{B}$ and $f \in \mathrm{Hom}_{\mathfrak{D}}[L, B_1]$, then there exists a unique $f^* : \hat{B}(L) \to B_1$ such that $f^* \circ i(L) = f$. Then $\hat{B} : \mathfrak{D} \to \mathfrak{B}$ can be extended to a reflector functor. L^* will be identified with

L, and $x \in \hat{B}(L)$ if and only if

$$x = a + \sum_{i=1}^{n} a_i \bar{b}_i + \bar{b},$$

where $a, a_i, b_i, b \in L$ and $\bar{b}_i$ is the complement of b_i in $\hat{B}(L)$.

1. Characterization of a relatively complemented distributive lattice.

THEOREM 1.1. *If $L \in \mathfrak{R}$, then L is convex in $\hat{B}(L)$.*

Proof. Suppose $a \leqslant b \leqslant c$, where $a, c \in L$ and $b \in \hat{B}(L)$. Since $b \in \hat{B}(L)$, b can be written in the form

$$b = u + \sum_{i=1}^{n} u_i \bar{v}_i + \bar{v},$$

where $u, u_i, v_i, v \in L$ and $u, \bar{v}$ or $u_i \bar{v}_i$ cannot occur. Now $a \leqslant b$ implies

$$b = a + b = a + u + \sum_{i=1}^{n} u_i \bar{v}_i + \bar{v},$$

and $b \leqslant c$ implies

$$b = a + cu + \sum_{i=1}^{n} cu_i \bar{v}_i + c\bar{v}.$$

Thus we can assume

$$b = u + \sum_{i=1}^{n} u_i \bar{v}_i,$$

where u always occurs and $u_i \bar{v}_i$ can or cannot occur. If no $u_i \bar{v}_i$ occurs, then $b = u \in L$. Assume that

$$b = u + \sum_{i=1}^{n} u_i \bar{v}_i$$

and that $u_i \bar{v}_i$ occurs. Then $uv_i \leqslant v_i \leqslant u_i + v_i$ for every i and there exists $v_i' \in L$ such that v_i' is the complement of v_i in $[uv_i, u_i + v_i]$. Hence

$$v_i' = (u_i + v_i)\bar{v}_i + uv_i = u\bar{v}_i + uv_i$$

and

$$b = u + \sum_{i=1}^{n} u_i \bar{v}_i = u + \sum_{i=1}^{n} (u_i \bar{v}_i + uv_i) = u + \sum_{i=1}^{n} v_i' \in L.$$

LEMMA 1.2. *Let $B \in \mathfrak{B}$ and let $L \in \mathfrak{R}$ be a sublattice of B such that $0 \in L$ and $\hat{B}(L) = B$. Then L is a prime ideal of B.*

Proof. If $1 \in L$, then $L \in \mathfrak{B}$ implies $\hat{B}(L) = L$, so $L = B$. Assume $1 \notin L$. By Theorem 1.1, L is convex in $\hat{B}(L)$, and $0 \in L$, $1 \notin L$ imply L is a proper ideal of B.

We claim that L is a prime ideal of B. Indeed, given $a \in B-L$, we show that $\bar{a} \in L$. The assumption $B = \hat{B}(L)$ yields

$$a = u + \sum_{i=1}^{n} u_i \bar{v}_i + \bar{v},$$

where $u, u_i, v_i, v \in L$ and u, v or $u_i \bar{v}_i$ need not occur. If $u_i \bar{v}_i$ occurs, we get $0 \leqslant u_i \bar{v}_i \leqslant u_i$; $0, u_i \in L$ implies $u_i v_i \in L$ by convexity of L. Thus $a = u + \bar{v}$, where $u, v \in L$ and $\bar{a} = \bar{u}v$. Since $v \in L$, $\bar{a} = \bar{u}v \in L$ by convexity of L.

A similar argument yields

LEMMA 1.3. *Let $B \in \mathfrak{B}$ and let $L \in \mathfrak{R}$ be a sublattice of B such that $1 \in L$ and $\hat{B}(L) = B$. Then L is an ultrafilter of B.*

THEOREM 1.4. *Let $B \in \mathfrak{B}$.*

(a) *If P and F are a non-principal prime ideal and ultrafilter, respectively, such that $F \neq B-P$, then P, F and $P \cap F$ are relatively complemented convex sublattices of B and $\hat{B}(P) = \hat{B}(F) = \hat{B}(P \cap F) = B$.*

(b) *If $L \in \mathfrak{R}$ is a sublattice of B such that $\hat{B}(L) = B$, then $L = P \cap F$, where P is either a non-principal prime ideal of B or $P = B$ and F is either a non-principal ultrafilter of B or $F = B$.*

Proof. (a) P is obviously a relatively complemented lattice and, clearly, P is a sublattice of B such that $0 \in P$. Suppose $a \in B-P$. Then $\bar{a} \in P$ implies $\hat{B}(P) = B$. Thus P is convex in B by Theorem 1.1.

Similarly, F is convex in B.

Now consider $P \cap F$. Clearly, $P \cap F$ is a relatively complemented sublattice of B. We show that $\hat{B}(P \cap F) = B$. Let $a \in B - P \cap F$. Then there are the following three possibilities:

(i) $a \notin P$ and $a \notin F$. Then $\bar{a} \in P \cap F$ and $a = \bar{\bar{a}} \in \hat{B}(P \cap F)$.

(ii) $a \notin P$ and $a \in F$. Then, for any $b \in P \cap F$, we have $ab \in P \cap F$ and $\bar{a} + b \in P \cap F$. Thus $a\bar{b} = \overline{\bar{a}+b} \in \hat{B}(P \cap F)$ and $ab \in P \cap F$ implies

$$a = ab + a\bar{b} \in \hat{B}(P \cap F).$$

(iii) $a \in P$ and $a \notin F$ — dual to (ii).

Thus $\hat{B}(P \cap F) = B$ and, by Theorem 1.1, $P \cap F$ is convex in B.

(b) If $0 \in L$, then L is a prime ideal by Lemma 1.2. If $1 \in L$, then L is an ultrafilter of B by Lemma 1.3.

Suppose $0 \notin L$ and $1 \notin L$. Then $\hat{B}(L) = B$, so that L is convex in B by Theorem 1.1. Let

$$P = \{x \in B : x \leqslant a \text{ for some } a \in L\}.$$

Then P is a proper ideal of B, and $P \supseteq L$ implies $\hat{B}(P) = B$. Hence P is a prime ideal of B by Lemma 1.2. Dually,

$$F = \{x \in B : x \geqslant a \text{ for some } a \in L\}$$

is an ultrafilter of B. Now $L \subseteq F$ and $L \subseteq P$ imply $L \subseteq P \cap F$. Let $x \in P \cap F$. Then there exist $a \in L$ such that $a \leqslant x$ and $b \in L$ such that $x \leqslant b$. Since L is convex in B, $x \in L$.

COROLLARY 1.5. *Every lattice $L \in \mathfrak{R}$ can be imbedded in a Boolean lattice B so that $L = P \cap F$, where P is a prime ideal of B and F is an ultrafilter of B.*

Proof. The proof follows if we take $B = \hat{B}(L)$ and apply Theorem 4 (b).

2. The free relatively complemented extension of a distributive lattice.

LEMMA 2.1. *If $L \in \mathfrak{D}$ is a convex sublattice of $B \in \mathfrak{B}$, then $L \in \mathfrak{R}$.*

Proof. Suppose $a, b, c \in L$ and $a \leqslant b \leqslant c$. Then there exists $b' \in B$ such that $b + b' = c$ and $bb' = a$. Since $a \leqslant b' \leqslant c$, $b' \in L$ by convexity of L. Thus $L \in \mathfrak{R}$.

THEOREM 2.2. *Let $L \in \mathfrak{D}$ and consider L as a sublattice of $\hat{B}(L)$. Then*

$$R(L) = \left\{x \in \hat{B}(L) : x \in L \text{ or } x = \sum_{i=1}^{n} a_i \bar{b}_i, \text{ where } a, a_i, b_i \in L\right\}$$

is the smallest relatively complemented sublattice of $\hat{B}(L)$ that contains L as a sublattice.

Proof. First we show that $R(L)$ is a relatively complemented lattice. Clearly, $R(L)$ is a lattice. Suppose $x \leqslant y \leqslant z$, where $x, z \in R(L)$ and $y \in \hat{B}(L)$. Then

$$x = a + \sum_{i=1}^{n} b_i \bar{c}_i \quad \text{and} \quad z = e + \sum_{j=1}^{n} f_j \bar{g}_j,$$

where $a, b_i, c_i, e, f_j, g_j \in L$ and $b_i \bar{c}_i$, $f_j \bar{g}_j$ cannot occur. Since $y \in \hat{B}(L)$,

$$y = u + \sum_{k=1}^{n} u_k \bar{v}_k + \bar{v} \quad \text{for } u, u_k, v_k, v \in L,$$

where u, $u_k \bar{v}_k$ or $\bar{v}$ cannot occur. If $a \in L$ and $a \leqslant x$, then $a \leqslant y$, so $a + y = y$. Similarly, since $yz = y$, we have

$$y = u'' + \sum_{k=1}^{n} u'_k \bar{v}'_k,$$

where u'', u'_k, $v'_k \in L$, and u'' must occur and $u'_k \bar{v}'_k$ can or cannot occur.

Therefore, $y \in R(L)$. Thus $R(L)$ is convex in $\hat{B}(L)$ and $R(L) \in \mathfrak{R}$ by Lemma 2.1.

Next we show that $R(L)$ is the smallest relatively complemented distributive lattice containing L as a sublattice.

Let $L' \in \mathfrak{R}$ be a sublattice of $\hat{B}(L)$ containing L. We show that if

$$x = a + \sum_{i=1}^{n} b_i \bar{c}_i \quad \text{for } a, a_i, b_i \in L,$$

then $x \in L'$. It suffices to show that, for any $a, b, c \in L$, we have $a + b\bar{c} \in L'$.

Now $a \leqslant a + c \leqslant a + b + c$ and $a, a + c, a + b + c \in L'$. Since $L' \in \mathfrak{R}$, there exists $c' \in L'$ such that $(a + c)c' = a$ and $a + c + c' = a + b + c$. But $c' \in \hat{B}(L)$ implies that

$$c' = a + (a + b + c)\overline{(a + c)} = a + (a + b + c)\bar{a}\bar{c} = a + b\bar{a}\bar{c} = a + b\bar{c} \in L'.$$

Note. $R(L)$ contains 0 if and only if L contains 0; similarly for 1.

Definition. Let $B \in \mathfrak{B}$ and let $L \in \mathfrak{D}$ be a sublattice of B. The *convex hull* of L in B, denoted by L_B^*, is the smallest convex sublattice of B containing L.

THEOREM 2.3. *If $L \in \mathfrak{D}$, then $L^*_{\hat{B}(L)} = R(L)$.*

Proof. Clearly, $\hat{B}(R(L)) = \hat{B}(L)$ since $L \subseteq R(L) \subseteq \hat{B}(R(L))$. Thus, by Theorem 1.1, $R(L)$ is convex in $\hat{B}(L)$, and $L^*_{\hat{B}(L)}$ is a sublattice of $R(L)$.

Also, $L^*_{\hat{B}(L)} \in \mathfrak{R}$, and L is a sublattice of $L^*_{\hat{B}(L)}$. Hence $R(L)$ is a sublattice of $L^*_{\hat{B}(L)}$ by Theorem 2.2.

Definition. Let $L \in \mathfrak{D}$. Then $\langle R(L), \lambda(L) \rangle$ is the *free relatively complemented extension* of L if $R(L) \in \mathfrak{R}$, $\lambda(L) : L \to R(L)$ is an imbedding map and whenever $M \in \mathfrak{R}$ and $f \in \mathrm{Hom}_{\mathfrak{D}}[L, M]$, then there exists a unique $f^* \in \mathrm{Hom}_{\mathfrak{R}}[R(L), M]$ such that $f^* \circ \lambda(L) = f$.

THEOREM 2.4. *Let $L \in \mathfrak{D}$ and let $\lambda(L) : L \to R(L)$ be the imbedding map. Then $\langle R(L), \lambda(L) \rangle$ is the free relatively complemented extension of L.*

Proof. Let $M \in \mathfrak{R}$ and let $f \in \mathrm{Hom}_{\mathfrak{D}}[L, M]$. Then

$$\hat{B}(f) \in \mathrm{Hom}_{\mathfrak{B}}[\hat{B}(L), \hat{B}(M)]$$

is the unique extension of f to $\hat{B}(L)$. Let $f^* = \hat{B}(f)|_{R(L)}$. We claim that $\mathrm{Im} f^* \subseteq M$. Indeed, let $x \in R(L)$. If $x \in L$, then $f^*(x) = f(x) \in M$. Now suppose

$$x = b + \sum_{i=1}^{n} c_i \bar{d}_i, \quad \text{where } b, c_i, d_i \in L.$$

Then

$$f^*(x) = f^*\left(b + \sum_{i=1}^{n} c_i \bar{d}_i\right) = f^*(b) + \sum_{i=1}^{n} f^*(c_i) f^*(\bar{d}_i)$$

$$= f(b) + \sum_{i=1}^{n} f(c_i)\overline{f(d_i)} \in R(f(L)).$$

Also, $f(L) \subseteq M$ implies $R(f(L)) \subseteq M$ by Theorem 2.2. Thus $\mathrm{Im} f^* \subseteq M$.

Clearly, $f^* \in \mathrm{Hom}_{\mathfrak{D}}[R(L), M]$ and $f^* \circ \lambda(L) = f$. The uniqueness of f^* follows from the uniqueness of $\hat{B}(f)$.

Remarks. (1) By Theorem 2.4, $R : \mathfrak{D} \to \mathfrak{R}$ can be extended to a reflector functor.

(2) It is well known (see [1]) that the free distributive lattice on n generators, n being finite, has the length $2^n - 2$. If $L \in \mathfrak{D}$, $|L| < \aleph_0$ and C is a maximal chain of L, then $\hat{B}(L) = \hat{B}(C)$. If the length of L is k, then $|\hat{B}(L)| = 2^k$.

(3) Let $\mathfrak{A}$ and $\mathfrak{C}$ be categories that have free objects. Let $F : \mathfrak{A} \to \mathfrak{C}$ be a reflector functor. Then it is known (see [2]) that if S is a set and A_S is free on S in $\mathfrak{A}$, then $F(A)_S$ is free on S in $\mathfrak{C}$.

From Remarks (2) and (3) we get the following characterization of the free objects in $\mathfrak{R}$.

THEOREM 2.5. *Let S be a non-empty set. If F_S is the free object on S in $\mathfrak{D}$, then $R(F_S)$ is the free object on S in $\mathfrak{R}$. If $|S| = n$, then $R(F_S) \in \mathfrak{B}$ and $|R(F_S)| = 2^{2^n - 2}$.*

REFERENCES

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