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SOME PROBLEMS CONCERNING CURVES

BY

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We use the term *curve* to mean any one-dimensional connected compact metric space. The aim of this paper is to present a number of problems which have been raised during a research conducted by the author(1). The background and the motivation to these problems come from the topological classification of curves and their set-theoretical properties closely related to the concept of connectedness. By a *mapping* we mean any continuous function from a topological space into another topological space.

For a metric space X, the *span* $\sigma(X)$ of X is defined to be the least upper bound of real numbers $r \geqslant 0$ satisfying the following condition: there exists a connected space Y and a pair of mappings $f, g\colon Y \to X$ such that $f(Y) = g(Y)$ and

$$\operatorname{dist}[f(y), g(y)] \geqslant r$$

for $y \in Y$. Thus we always have $\sigma(X) \geqslant 0$. The space R of real numbers has span $\sigma(R) = \infty$, and the space S of complex numbers with module one has span $\sigma(S) = \operatorname{diam} S = 2$. If A is an *arc*, i.e. a curve homeomorphic to a segment of R, then $\sigma(A) = 0$. By a *tree* we mean any curve homeomorphic to a one-dimensional polyhedron which contains no topological copy of the circle S. It is not difficult to prove that all trees of span zero are arcs (see [9], p. 200). A curve X is said to be *arc-like* (*tree-like*) provided, for each $\varepsilon > 0$, there exists a mapping $f\colon X \to Y$ of X into an arc (a tree) Y such that

$$\operatorname{diam} f^{-1}(y) < \varepsilon$$

for $y \in Y$. Clearly, all subcurves of arc-like or tree-like curves are arc-like or tree-like, respectively. It is known that all arc-like curves have span zero (see [9], p. 210).

PROBLEM 1. *Let X be a curve such that $\sigma(X) = 0$. Is X arc-like?* (**P 717**)

(1) The problems were compiled when the author was visiting the Middle East Technical University, Ankara, Turkey, in Spring Semester of the academic year 1969 - 1970.

For a tree-like curve X, the *width* $w(X)$ of X has been defined by C. E. Burgess in terms of metric properties of some standard covers of X (see [2], p. 447). It seems very likely that there exists a relation between the span and the width of tree-like curves. Let us denote by 2_c^X the collection of all subcurves of X.

PROBLEM 2. *Suppose X is a tree-like curve. Is it true that*

$$\sigma(X) = \operatorname{Sup}\{w(Y):\ Y \in 2_c^X\}\,? \quad (\mathbf{P\ 718})$$

In the case Problem 2 has an affirmative solution, one could guess a negative answer to Problem 1 (see [3], p. 479). We recall that a mapping is called *open* provided the images of open sets under the mapping are open.

PROBLEM 3. *Suppose X is a curve such that X is the image of an arc-like curve under an open mapping. Is X arc-like?* (**P 719**)

For curves X and Y, a mapping $f\colon X \to Y$ is called *confluent* provided, for each subcurve K of Y and each connected component C of $f^{-1}(K)$, we have $f(C) = K$. It is known that all open mappings as well as all monotone mappings which transform the space onto the space are confluent (see [11], p. 223).

PROBLEM 4. *Suppose X is a curve such that X is the image of an arc-like curve under a confluent mapping. Is X arc-like?* (**P 720**)

PROBLEM 5. *Suppose X is a curve such that X is the image of a tree-like curve under a confluent mapping. Is X tree-like?* (**P 721**)

An affirmative answer to Problem 4 would, of course, imply an affirmative answer to Problem 3. It is known that if a curve X is the image of an arc-like curve under a monotone mapping, then X is arc-like (see [1], p. 47). Recall that a curve is said to be *decomposable* provided it admits a decomposition into two proper subcurves. There exist indecomposable curves (see [8], p. 204) and it has been proved that Problem 4 reduces to such curves (see [7], p. 389). A partial solution of Problem 5 has been obtained too. Namely, a curve is said to be *unicoherent* provided, for each decomposition into two subcurves, their intersection is connected. A curve X is called *hereditarily unicoherent* (*hereditarily decomposable*) provided each subcurve of X is unicoherent (decomposable). It is known that all tree-like curves are hereditarily unicoherent, and all images of hereditarily unicoherent and hereditarily decomposable curves under confluent mappings are hereditarily unicoherent and hereditarily decomposable (see [5], p. 217). Moreover, all hereditarily unicoherent and hereditarily decomposable curves are tree-like (see [6], p. 20). A curve X is said to be *acyclic* provided each mapping from X into S is homotopic to a constant mapping. It is known that all tree-like curves are acyclic, and there exists an acyclic curve which is not tree-like (see [4], p. 81).

Also, all subcurves of acyclic curves are acyclic and unicoherent (see [8], p. 354 and 437). It has been proved that all images of acyclic curves under confluent mappings are acyclic (see [11], p. 230).

For a topological space X, the *quasi-component* $Q(X, x)$ of X at a point $x \in X$ is the intersection of all closed-open subsets of X that contain x. A space is said to be *totally disconnected* provided each of its quasi-components is degenerate. A curve is called *rational* provided it admits an open basis consisting of sets with countable boundaries. We now want to find a relation between the rationality of a curve and the existence of a totally disconnected subset of it.

LEMMA. *If P is a proper subset of a compact metric space X such that each quasi-component of P is zero-dimensional and locally compact, then* $\dim P \leqslant \dim(X \setminus P)$.

Proof. In the case of compact P, the quasi-components of P coincide with connected components of P, thus they are all degenerate and P is totally disconnected and zero-dimensional. Hence, in this case, the required inequality trivially holds, and we can assume that P is non-compact. Let f be a mapping of P into the Cantor set such that $f^{-1}f(x) = Q(P, x)$ for $x \in P$ (see [8], p. 148). It follows that

$$\dim P \leqslant \dim f(P) + \operatorname{Max}\{\dim f, \operatorname{def} P\} = \operatorname{Max}\{0, \operatorname{def} P\},$$

where $\operatorname{def} P$ denotes the minimum dimension of remainders in compactifications of P (see [10], p. 225). Since P is non-compact, we have

$$0 \leqslant \operatorname{def} P \leqslant \dim(X \setminus P)$$

which completes the proof of the lemma.

THEOREM. *If X is a curve, then the following conditions are equivalent to each other*:

(i) *X is rational,*

(ii) *$X = P \cup Q$, where P is zero-dimensional and Q is countable,*

(iii) *$X = P \cup Q$, where P is totally disconnected and Q is countable,*

(iv) *$X = P \cup Q$, where P has zero-dimensional locally compact quasi-components and Q is countable.*

Proof. The equivalence of (i) and (ii) is well-known (see [8], p. 285). Since all zero-dimensional sets are totally disconnected, (ii) implies (iii). Clearly, (iii) implies (iv). If (iv) holds, then $X \setminus P \subset Q$ is zero-dimensional and, by the lemma, so is P. Thus we get (ii) and the proof of the theorem is complete.

For a separable metric space X and a countable ordinal α, the *quasi-component of order α* $Q^\alpha(X, x)$ of X at a point $x \in X$ is defined inductively in the following way. We put $Q^0(X, x) = X$ and define

$$Q^{\alpha+1}(X, x) = Q[Q^\alpha(X, x), x], \qquad Q^\lambda(X, x) = \bigcap_{\alpha<\lambda} Q^\alpha(X, x)$$

where λ is a limit ordinal. Thus quasi-components are quasi-components of order 1 and we have a decreasing transfinite sequence

$$Q^0(X, x) \supset Q^1(X, x) \supset \ldots \supset Q^\alpha(X, x) \supset \ldots$$

which consists of closed subsets of X. Consequently, there exists a countable ordinal α such that $Q^\alpha(X, x)$ coincides with $Q^{\alpha+1}(X, x)$, which means that $Q^\alpha(X, x)$ is connected. The countable ordinal

$$nc(X, x) = \operatorname{Min}\{\alpha\colon Q^\alpha(X, x) = Q^{\alpha+1}(X, x)\}$$

is called the *non-connectivity index* of the space X at the point x. Clearly, the quasi-component of order $nc(X, x)$ of X at x is equal to the connected component of X to which x belongs. Let Ω denote the least uncountable ordinal. It is known (see [12], p. 367) that if X is a rational curve and $A \subset X$, then

$$\operatorname{Sup}\{nc(A, x)\colon x \in A\} < \Omega.$$

PROBLEM 6. *Suppose X is a rational curve. Is it true that*

$$(*) \qquad \operatorname{Sup}\{nc(A, x)\colon A \subset X, x \in A\} < \Omega\,? \quad (\mathbf{P\ 722})$$

PROBLEM 7. *Suppose X is a rational curve and $A \subset X$. Is it true that the collection of all non-degenerate quasi-components of order* 1 *of A is countable?* (**P 723**)

We say that a curve is *Suslinian* provided each collection of pairwise disjoint subcurves of it is countable. It is known that all rational curves are Suslinian, all Suslinian curves are hereditarily decomposable, and there exists a hereditarily unicoherent and arcwise connected Suslinian curve([2]) which is not rational (see [13], p. 135).

PROBLEM 8. *Suppose X is a Suslinian curve. Does* (*) *hold?* (**P 724**)

PROBLEM 9. *Suppose X is a Suslinian curve and $A \subset X$. Is it true that the collection of all non-degenerate quasi-components of orders $\alpha < \Omega$ of A is countable?* (**P 725**)

Let us note that positive solutions of Problems 8 and 9 would imply positive solutions of Problems 6 and 7, respectively. The class of Suslinian curves seems, however, much harder to deal with than the class of rational curves. This can be seen when one tries to prove a decomposition property as motivated by the theorem above. We say that a space X is *hereditarily disconnected* (*hereditarily discontinuous*) provided each connected (connected compact) subset of X is degenerate. Thus all to-

([2]) In an abbreviated terminology, hereditarily unicoherent and arcwise connected curves are called *dendroids*, while hereditarily unicoherent and hereditarily decomposable curves are called *λ-dendroids*. All dendroids are λ-dendroids.

tally disconnected spaces are hereditarily disconnected, and all hereditarily disconnected spaces are hereditarily discontinuous. It is known that if X is a hereditarily unicoherent Suslinian curve, then there exists a decomposition $X = P \cup Q$ where P is hereditarily discontinuous and Q is countable (see [13], p. 133).

PROBLEM 10. *Suppose X is a Suslinian curve. Does there exist a decomposition $X = P \cup Q$ where P is hereditarily discontinuous and Q is countable?* (**P 726**)

PROBLEM 11. *Suppose X is a hereditarily unicoherent Suslinian curve. Does there exist a decomposition $X = P \cup Q$ where P is hereditarily disconnected and Q is countable?* (**P 727**)

PROBLEM 12. *Suppose X is a hereditarily unicoherent Suslinian curve. Does there exist a countable subset $A \subset X$ such that $X \setminus A$ is not connected?* (**P 728**)

Let us point out that there exists a locally connected Suslinian curve K such that K lies on the plane and, for each countable subset $A \subset K$, the set $K \setminus A$ is connected (see [15], p. 337). It readily follows that K is neither unicoherent nor rational. The notion of rationality for curves has also been investigated in another direction. By the *rim-type* of a rational curve X we mean the minimum ordinal α such that X admits an open basis consisting of sets with countable boundaries whose α-th derivatives are empty. Hence rim-types of rational curves are countable ordinals, and an arc is the simplest example of a curve of rim-type 1. It is known that all hereditarily unicoherent rational curves of finite rim-types contain arcs (see [14]).

PROBLEM 13. *Suppose X is a rational curve of finite rim-type n and $m = 1, \ldots, n$. Does there exist a subcurve $Y \subset X$ such that Y has rim-type m?* (**P 729**)

We say that a curve X is *radial* provided there exist a point $p \in X$ and a collection $\boldsymbol{A}$ of arcs such that p is an end point of each arc from $\boldsymbol{A}$, the union of all arcs from $\boldsymbol{A}$ is X, and $A_1 \cap A_2 = \{p\}$ for each two arcs $A_1, A_2 \in \boldsymbol{A}$. Clearly, all radial curves are arcwise connected. It is known that there exists a radial hereditarily decomposable curve which contains a topological copy of the circle (see [14]). However, this curve is not locally connected.

PROBLEM 14. *Suppose X is a radial curve such that X is hereditarily decomposable and locally connected. Is X unicoherent?* (**P 730**)

PROBLEM 15. *Let X be a radial curve such that X contains no topological copy of the product of an arc with the Cantor set. Is X rational?* (**P 731**)

Added in proof. A negative answer has been recently given to Problem 2 (W. T. Ingram) as well as to Problems 11 and 12 (H. Cook).

Arc-like rational curves have been constructed to provide, for $m > 1$, a negative solution of Problem 13 (B. B. Epps, Jr.); it remains unsolved for $m = 1$. Also, there has been obtained an affirmative solution of Problem 5 (B. McLean).

REFERENCES

[1] R. H. Bing, *Concerning hereditarily indecomposable continua*, Pacific Journal of Mathematics 1 (1951), p. 43 - 51.

[2] C. E. Burgess, *Collections and sequences of continua in the plane II*, Pacific Journal of Mathematics 11 (1961), p. 447 - 454.

[3] C. E. Burgess, *Continua which have width zero*, Proceedings of the American Mathematical Society 13 (1962), p. 477 - 481.

[4] J. H. Case and R. E. Chamberlin, *Characterizations of tree-like continua*, Pacific Journal of Mathematics 10 (1960), p. 73 - 84.

[5] J. J. Charatonik, *Confluent mappings and unicoherence of continua*, Fundamenta Mathematicae 56 (1964), p. 213 - 220.

[6] H. Cook, *Tree-likeness of dendroids and λ-dendroids*, Fundamenta Mathematicae 68 (1970), p. 19 - 22.

[7] J. B. Fugate, *A characterization of chainable continua*, Canadian Journal of Mathematics 21 (1969), p. 383 - 393.

[8] K. Kuratowski, *Topology*, vol. II, Academic Press 1968.

[9] A. Lelek, *Disjoint mappings and the span of spaces*, Fundamenta Mathematicae 55 (1964), p. 199 - 214.

[10] — *Dimension and mappings of spaces with finite deficiency*, Colloquium Mathematicum 12 (1964), p. 221 - 227.

[11] — *On confluent mappings*, Colloquium Mathematicum 15 (1966), p. 223 - 233.

[12] — *On the topology of curves I*, Fundamenta Mathematicae 67 (1970), p. 359 - 367.

[13] — *On the topology of curves II*, Fundamenta Mathematicae 70 (1971), p. 131 - 138.

[14] — and L. Mohler, *On the topology of curves III*, Fundamenta Mathematicae 71 (1971).

[15] S. Mazurkiewicz, *Sur les courbes d'ordre* $\mathfrak{c}$, Fundamenta Mathematicae 16 (1930), p. 337 - 347.

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