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NOTE ON SUMMABILITY (L) OF FOURIER INTEGRALS

BY

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1. Let $f(t)\in L(0, T)$ for any $T > 0$. Nayak [2] has defined summability (L) for an integral.

We say that the integral $\int_0^\infty f(t)\,dt$ is *summable* (L) to s if

$$\lim_{\delta\to 0}\frac{1}{\log(1/\delta)}\int_1^\infty \frac{e^{-\delta u}}{u}F(u)\,du = s, \qquad \text{where } F(u) = \int_0^u f(t)\,dt.$$

In this note, we intend to generalize the result on summability (L) of the Fourier Integral

$$\int_0^\infty \psi(u)\,du, \qquad \text{where } \psi(u) = \frac{1}{\pi}\int_{-\infty}^{+\infty} f(t)\cos u(x-t)\,dt.$$

Let us write

$$\varphi(t) = f(x+t)+f(x-t)-2s,$$

$$\Psi(t) = \int_0^t \psi(u)\,du,$$

$$\varphi_a(t) = \frac{a}{t^a}\int_0^t (t-u)^{a-1}\varphi(u)\,du \qquad (a > 0),$$

$$\varphi_0(t) = \varphi(t),$$

$$H_a(t) = \int_t^1 \frac{\varphi_a(u)}{u}\,du \qquad (a \geqslant 0),$$

$$M(t) = \int_1^\infty \frac{e^{-u}}{u}\sin ut\,du,$$

$$N_a(t) = t^{a+1}\left(\frac{d}{dt}\right)^a \frac{M(t)}{t} \qquad (a > 0),$$

$$N_0(t) = M(t).$$

2. In an attempt to study the summability (L) of a Fourier integral, Nayak [2] has proved the following

THEOREM A. *If*

$$(2.1)\qquad H_0(t) \equiv \int_t^1 \frac{\varphi(u)}{u}\,du = o\left(\log\frac{1}{t}\right) \quad (t\to+0),$$

then the integral

$$(2.2)\qquad \int_0^\infty \psi(u)\,du$$

is summable (L) *to the value* s.

The object of this note is to prove summability (L) of the Fourier integral under a less stringent condition imposed on the generating function. In fact we shall establish the following

THEOREM. *If*

$$(2.3)\qquad H_\alpha(t) \equiv \int_t^1 \frac{\varphi_\alpha(u)}{u}\,du = o\left(\log\frac{1}{t}\right) \quad (\alpha\geqslant 0),$$

then the integral (2.1) *is summable* (L) *to the value* s.

To show that our hypothesis (2.3) is weaker than that of Nayak, we quote the following theorem by Nanda and Das [1]:

THEOREM B. *Let* $\beta>\alpha\geqslant 0$, *then*

$$H_\alpha(t) = o\left(\log\frac{1}{t}\right) \text{ implies } H_\beta(t) = o\left(\log\frac{1}{t}\right) \quad (t\to+0).$$

3. We shall require the following lemmas for the proof of the theorem.

LEMMA 1. *Let* $\alpha = 0,1,2,\ldots$ *Then*

$$\begin{aligned}&(3.1)\\&(3.2)\end{aligned}\qquad \frac{d}{dt}\big(N_\alpha(t)\big) = \begin{cases} O(t^\alpha/\delta^{\alpha+1}), \\ O(\delta/t^2)+O(1) \quad (t>\delta).\end{cases}$$

Proof. We have

$$(3.3)\qquad \frac{d}{dt}N_\alpha(t) = t^\alpha\left\{(\alpha+1)\left(\frac{d}{dt}\right)^\alpha \frac{M(t)}{t} + t\left(\frac{d}{dt}\right)^{\alpha+1}\frac{M(t)}{t}\right\}$$

$$= t^\alpha\left(\frac{d}{dt}\right)^{\alpha+1} M(t).$$

It is easy to see that

$$\left(\frac{d}{dt}\right)^{\alpha+1} M(t) = \int_1^\infty \frac{e^{-\delta u}}{u}\left(\frac{\partial}{\partial t}\right)^{\alpha+1} \sin ut\,dt.$$

Now,

$$\left(\frac{\partial}{\partial t}\right)^{a+1} \sin ut = O(u^{a+1}).$$

Thus

$$\left(\frac{d}{dt}\right)^{a+1} M(t) = O\left(\int_0^\infty e^{-\delta u} u^a du\right) = O(\delta^{-a-1}).$$

In view of (3.3), this completes the proof of (3.1).

To prove (3.2), we write

$$T_a = \left(\frac{d}{dt}\right)^{a+1} M(t),$$

and we shall prove the estimate by induction on a.

We find that for $a = 0$ ($\delta < t$),

$$T_0 = \frac{\partial}{\partial t} \int_1^\infty \frac{e^{-\delta u}}{u} \sin ut\, du$$

$$= \frac{e^{-\delta}\{\delta\cos t - t\sin t\}}{\delta^2 + t^2} = O\left(\frac{\delta}{t^2}\right) + O(1).$$

Let us write $g = g(\delta, t) = (\delta^2 + t^2)e^{\delta}$. Hence $T_0 g = \delta\cos t - t\sin t$.

Let us suppose that Lemma 1 holds for $a \leqslant k-1$, $k = 1, 2, \ldots$ Then

$$\left(\frac{\partial}{\partial t}\right)^k (T_0 g) = \left(\frac{\partial}{\partial t}\right)^k (\delta\cos t - t\sin t).$$

By Leibniz theorem,

$$T_k g + \binom{k}{1} T_{k-1} g_1 + \ldots + T_0 g_k = (\delta + k)\cos(t + \tfrac{1}{2}k\pi) - t\sin(t + \tfrac{1}{2}k\pi).$$

Hence,

$$|T_k| =$$

$$= \left| -\frac{1}{g}\left\{\binom{k}{1} T_{k-1} 2te^{\delta} + \binom{k}{2} T_{k-2} 2e^{\delta} - (\delta + k)\cos(t + \tfrac{1}{2}k\pi) + t\sin(t + \tfrac{1}{2}k\pi)\right\}\right|.$$

Since $1/g = O(1/t^2)$, we have

$$|T_k| = O\left(\frac{\delta}{t^{k+1}} \frac{t}{t^2}\right) + O\left(\frac{\delta}{t^k} \frac{1}{t^2}\right) + O(1) = O\left(\frac{\delta}{t^{2+k}}\right) + O(1).$$

Now (3.2) follows.

LEMMA 2. *For fixed δ and $k = 0, 1, \ldots,$*

$$\left(\frac{\partial}{\partial t}\right)^k \frac{M(t)}{t} = O(1) \quad (t \to +0).$$

Proof. Since

$$N_k(t) = t^{k+1}\left(\frac{\partial}{\partial t}\right)^k \frac{M(t)}{t},$$

we have

$$\left(\frac{\partial}{\partial t}\right)^k \frac{M(t)}{t} = \frac{N_k(t)}{t^{k+1}}.$$

Thus

$$\lim_{t\to+0}\left|\left(\frac{\partial}{\partial t}\right)^k \frac{M(t)}{t}\right| = \lim_{t\to+0}\left|\frac{\frac{\partial}{\partial t}(N_k(t))}{(k+1)t^k}\right| = O(1)$$

for fixed δ, when $t \to +0$, by (3.1).

LEMMA 3. *For fixed t $(0 < t < 1)$ and $k = 0, 1, \ldots,$*

$$\left(\frac{\partial}{\partial t}\right)^k \frac{M(t)}{t} = O(1) \quad \text{as } \delta \to 0.$$

Moreover,

$$\left(\frac{\partial}{\partial t}\right)^k \frac{M(t)}{t} \to (-1)^k k! \frac{\pi}{2t^{k+1}} + O(1) \quad \text{as } \delta \to 0.$$

Proof. We have

$$\frac{M(t)}{t} = \frac{1}{t}\int_0^\infty e^{-\delta u}\frac{\sin ut}{u}\,du - \int_0^1 e^{-\delta u}\frac{\sin ut}{ut}\,du$$

$$= \frac{1}{t}\tan^{-1}\left(\frac{t}{\delta}\right) - \sum_{j=0}^{\infty}\frac{(-1)^j}{(2j+1)!}t^{2j}\int_0^1 e^{-\delta u}u^{2j}\,du,$$

since the series is uniformly convergent.

Now

$$\lim_{\delta\to 0}\left(\frac{\partial}{\partial t}\right)^k \frac{M(t)}{t}$$

$$= \lim_{\delta\to 0}\left(\frac{\partial}{\partial t}\right)^k\left(\frac{1}{t}\tan^{-1}\left(\frac{t}{\delta}\right)\right) - \lim_{\delta\to 0}\left(\frac{\partial}{\partial t}\right)^k \sum_{j=0}^{\infty}\frac{(-1)^j}{(2j+1)!}t^{2j}\int_0^1 e^{-\delta u}u^{2j}\,du$$

$$\equiv T_1 + T_2.$$

$1/t$ and $\tan^{-1}t/\delta$ are differentiable any number of times in $0 < t < 1$ and $\delta \neq 0$ and hence,

$$T_1 = \lim_{\delta\to 0}\left(\frac{\partial}{\partial t}\right)^k\left(\frac{1}{t}\tan^{-1}\frac{t}{\delta}\right) = \left(\frac{\partial}{\partial t}\right)^k\left(\frac{1}{t}\lim_{\delta\to 0}\tan^{-1}\frac{t}{\delta}\right) = (-1)^k k!\frac{\pi}{2t^{k+1}}.$$

Further,

$$|T_2| = \lim_{\delta\to 0}\left|\sum_{j=0}^{\infty}\frac{(-1)^j(2j)(2j-1)\dots(2j-k+1)}{(2j+1)!}t^{2j-k}\int_0^1 e^{-\delta u}u^{2j}du\right| = O(1),$$

since

$$\sum_{j=0}^{\infty}\frac{(2j)\dots(2j-k+1)}{(2j+1)!}t^{2j-1} \leqslant \sum_{j=1}^{\infty}t^{2j-1},$$

and

$$\int_0^1 e^{-\delta u}u^{2j}du \leqslant 1.$$

4. Proof of the Theorem. We remark that there is no loss of generality in assuming a to be integral, in view of Theorem B. In what follows, we assume a to be integral.

We have,

$$\begin{aligned}\psi(u) &= \frac{1}{\pi}\int_1^{\infty}[f(x+t)+f(x-t)]\cos ut\,dt + \\ &\quad + \frac{2s}{\pi}\int_0^1 \cos ut\,dt + \frac{1}{\pi}\int_0^1 \varphi(t)\cos ut\,dt \\ &= \psi_1(u)+\psi_2(u)+\psi_3(u).\end{aligned}$$

Since (L) is a regular method and $f(t)\in L(-\infty,\infty)$,

$$\int_0^{\infty}\psi_1(u)\,du \quad\text{and}\quad \int_0^{\infty}\psi_2(u)\,du$$

are summable (L), as in Nayak [2].

The summability (L) of $\int_0^{\infty}\psi_3(u)\,du$ is equivalent to showing

$$\int_1^{\infty}\frac{e^{-\delta u}}{u}\int_0^1\varphi(t)\frac{\sin ut}{t}\,dt\,du = o\left(\log\frac{1}{\delta}\right), \quad \text{as } \delta\to 0. \tag{4.1}$$

By Fubini's theorem, the integral on the left of (4.1) equals

$$\int_0^1 \frac{\varphi(t)}{t} M(t)\,dt. \tag{4.2}$$

Integrating the integral in (4.2) a times by parts we obtain

(4.3)

$$\left[\sum_{r=1}^{a} (-1)^{r-1} \frac{t^r}{r!} \varphi_r(t) \left(\frac{\partial}{\partial t}\right)^{r-1} \frac{M(t)}{t}\right]_0^1 + \frac{(-1)^a}{a!} \int_0^1 t^a \varphi_a(t) \left(\frac{\partial}{\partial t}\right)^a \frac{M(t)}{t}\,dt$$

$$\equiv L_1 + L_2.$$

Now,

$$\int_0^1 t^a \varphi_a(t) \left(\frac{\partial}{\partial t}\right)^a \frac{M(t)}{t}\,dt = -\int_0^1 N_a(t)\,dH_a(t)$$

$$= H_a(1) N_a(1) + \lim_{t\to+0} H_a(t) N_a(t) + \int_0^1 H_a(t) \frac{\partial}{\partial t} N_a(t)\,dt$$

$$= \left\{\int_0^\delta + \int_\delta^1\right\} H_a(t) \frac{\partial}{\partial t} N_a(t)\,dt \equiv I_1 + I_2.$$

By Lemma 1,

$$I_1 = \int_0^\delta o\left(\log\frac{1}{t}\right) O(t^a \delta^{-a-1})\,dt = o(\log 1/\delta).$$

Also

$$I_2 = \int_\delta^1 o\left(\log\frac{1}{t}\right) \{O(\delta t^{-2}) + O(1)\} = o(\log 1/\delta).$$

Thus $L_2 = o(\log 1/\delta)$. Since $t^r \varphi_r(t) = o(1)$ as $t \to 0$, we have, by Lemma 2 and Lemma 3,

$$\left[\sum_{r=1}^{a} (-1)^{r-1} \frac{t^r}{r!} \varphi_r(t) \left(\frac{\partial}{\partial t}\right)^{r-1} \frac{M(t)}{t}\right]_{t=0}^{t=1}$$

$$= \left[\sum_{r=1}^{a} (-1)^{r-1} \frac{t^r \varphi_r(t)}{r!} \left(\frac{\partial}{\partial t}\right)^{r-1} \frac{M(t)}{t}\right]_{t\to 1-}$$

$$= \sum_{r=1}^{a} (-1)^{2(r-1)} \frac{\varphi_r(1)}{r!} (r-1)! \frac{\pi}{2} + \sum_{r=1}^{a} \frac{\varphi_r(1)}{r!} O(1)$$

$$= o(\log 1/\delta) \quad \text{as } \delta \to 0.$$

Thus $L_1 = o(\log 1/\delta)$ and this completes the proof.

REFERENCES

[1] M. M. Nanda and G. Das, *Summability (L) of Fourier series*, Proceedings of the Cambridge Philosophical Society 67 (1970), p. 327-331.

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