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OPERATORS ON SOME FUNCTION SPACES

BY

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1. Introduction. The Gleason-Kahane-Żelazko theorem [2], [4], [10] states that if τ is a linear functional on a complex Banach algebra A such that

$$\tau(x) \in \sigma(x) \tag{1}$$

for every $x \in A$, where $\sigma(x)$ denotes the spectrum of x in A, then τ is multiplicative on A, i.e.,

$$\tau(xy) = \tau(x)\tau(y) \qquad (x, y \in A). \tag{2}$$

(See also Żelazko [11] and Rudin [7], p. 233.)

In [9], applying this interesting theorem, we have observed that if X is a compact Hausdorff space and T is a linear operator from $C(X)$ to $C(X)$, where $C(X)$ denotes the space of all complex-valued continuous functions on X, such that

$$|Tf| > 0 \quad \text{on } X \tag{3}$$

for every $f \in C(X)$ with $|f| > 0$ on X, then T has the form

$$Tf(x) = r(x)\cdot f(\varphi(x)) \qquad (f \in C(X),\ x \in X), \tag{4}$$

where $r \in C(X)$ and φ is a continuous mapping from X to X.

In the present paper we intend to apply the Gleason-Kahane-Żelazko theorem to obtain similar results for linear operators on $H_\infty(D)$, the space of all bounded analytic functions on the open unit disc $D = \{z: |z| < 1\}$ in the complex plane, and on $A(C)$, the space of all complex-valued continuous functions on the unit circle $C = \{z: |z| = 1\}$ in the complex plane that have absolutely convergent Fourier series.

2. Operators on $H_\infty(D)$. We recall that $H_\infty(D)$ is a complex Banach space with the norm

$$\|f\|_\infty = \sup_{z \in D} |f(z)| \qquad (f \in H_\infty(D)),$$

and that it is also a Banach algebra under the usual pointwise multiplication. If T is a linear operator from $H_\infty(D)$ to $H_\infty(D)$, we shall write

$$Z(T) = \{z \in D\colon Tf(z) = 0 \text{ for some invertible } f \in H_\infty(D)\}.$$

Here the continuity of T is not assumed. Our first result is the following

THEOREM 1. *Let T be a linear operator from $H_\infty(D)$ to $H_\infty(D)$. Then the following two statements* (I) *and* (II) *are equivalent*:

(I) *$Z(T)$ has no limit point in D and T is one-to-one (i.e., $Tf = 0$ implies $f = 0$).*

(II) *T has the form*

$$Tf(z) = h(z)\cdot f\big(a(z)\big) \qquad (f \in H_\infty(D),\, z \in D),$$

where $h, a \in H_\infty(D)$, h is not identically zero on D, a is not constant on D, and $\|a\|_\infty \leqslant 1$.

Proof. (I) implies (II). If we let $\Omega = D \cap Z(T)^c$, then Ω is open and connected. For $z \in \Omega$ and $f \in H_\infty(D)$, define

$$(5) \qquad \tau_z(f) = Tf(z)/T1(z).$$

τ_z is a linear functional on $H_\infty(D)$ satisfying

$$\tau_z(1) = 1 \quad \text{and} \quad \tau_z(f) \neq 0$$

for every invertible $f \in H_\infty(D)$. Therefore we may apply the Gleason-Kahane-Żelazko theorem to infer that τ_z is multiplicative on $H_\infty(D)$. Let the letter j denote the identity function: $j(z) = z$, and define a function a on Ω by the relation

$$a(z) = \tau_z(j) \qquad (z \in \Omega).$$

It follows that $|a(z)| \leqslant \|\tau_z\|\cdot\|j\|_\infty = 1$ $(z \in \Omega)$, and a is an analytic function on Ω by (5). Thus a may and will be regarded as an analytic function on D, because $Z(T)$ has no limit point in D. We now prove that $|a(z)| < 1$ for every $z \in D$. In fact, if $|a(z)| = 1$ for some $z \in D$, then the maximum modulus principle implies that $a = c$ on D, where c is a constant of absolute value 1, and so it follows from (5) that $Tj(z) = cT1(z)$ $(z \in D)$, which is impossible, since T is one-to-one. Hence it follows (cf. Hoffman [3], p. 160) that if $z \in \Omega$ and $f \in H_\infty(D)$, then $\tau_z(f) = f\big(a(z)\big)$. Thus

$$Tf(z) = h(z)\cdot f\big(a(z)\big) \qquad (f \in H_\infty(D),\, z \in D),$$

where we let $h = T1$. Since T is one-to-one, a cannot be constant on D. It is clear that $a \in H_\infty(D)$.

(II) implies (I). This is immediate from the open mapping theorem and the unicity theorem for analytic functions defined on an open and connected set in the complex plane.

COROLLARY 2 (cf. Nagasawa [5] and deLeeuw-Rudin-Wermer [1]). *Let T be a linear operator from $H_\infty(D)$ to $H_\infty(D)$. Assume that T is one-to-one and onto, and that $|Tf| > 0$ on D and $|T^{-1}f| > 0$ on D for every invertible $f \in H_\infty(D)$. If $\|T1\|_\infty = 1$ and $\|f\|_\infty \leqslant \|Tf\|_\infty$ for every $f \in H_\infty(D)$, or if $T1 = c$ on D, where c is a constant of absolute value 1, then T is an isometry.*

Proof. By Theorem 1, T has the form

$$Tf(z) = h(z)\cdot f\big(a(z)\big) \quad (f \in H_\infty(D),\ z \in D),$$

where $h \in H_\infty(D)$ is invertible and a is a conformal mapping from D onto D. If $\|T1\|_\infty = 1$ and $\|f\|_\infty \leqslant \|Tf\|_\infty$ for every $f \in H_\infty(D)$, then it follows that

$$\|1/T1\|_\infty = \|1/h\|_\infty = \|T^{-1}\| \leqslant 1,$$

and so $|T1| = 1$ on D, thus $T1 = c$ on D for some constant c of absolute value 1. Therefore T has the form $Tf(z) = c\cdot f\big(a(z)\big)$, and this completes the proof.

3. Operators on $A(C)$. We recall that $A(C)$ is a complex Banach space with the norm

$$\|f\| = \sum_{n=-\infty}^{\infty} |\hat{f}(n)| \quad (f \in A(C)),$$

where $\hat{f}$ is defined by

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\theta})e^{-in\theta}\,d\theta.$$

It is also a Banach algebra under the usual pointwise multiplication.

THEOREM 3. *Let T be a linear operator from $A(C)$ to $A(C)$. Assume that $f \in A(C)$ and $|f| > 0$ on C imply $|Tf| > 0$ on C. Then T has the form*

$$Tf(z) = h(z)\cdot f(cz^n) \quad (f \in A(C),\ z \in C),$$

where $h \in A(C)$ is invertible, c is a constant of absolute value 1, and n is an integer.

Proof. Write $h = T1$. Since $|h| > 0$ on C, $1/h \in A(C)$ by the inversion theorem of Wiener (see, for example, Rudin [8], p. 399). For $z \in C$ and $f \in A(C)$, let us define

$$\tau_z(f) = Tf(z)/h(z).$$

Then τ_z is a linear functional on $A(C)$ satisfying $\tau_z(1) = 1$ and $\tau_z(f) \neq 0$ for every invertible $f \in A(C)$. Hence τ_z is multiplicative on $A(C)$ by the Gleason-Kahane-Żelazko theorem, and thus there exists a complex number $a(z) \in C$ satisfying $\tau_z(f) = f\big(a(z)\big)$ for every $f \in A(C)$. Since

$$f\big(a(z)\big) = Tf(z)/h(z) \in A(C)$$

for every $f \in A(C)$, it follows from the Leibenson-Kahane theorem (cf. Rudin [6], p. 94) that a has the form

$$a(z) = c \cdot z^n \quad (z \in C),$$

where $|c| = 1$ and n is an integer, and the proof is complete.

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