COLLOQUIUM MATHEMATICUM

VOL. XXIX 1974 FASC. 2

# *NORMAL SOLVABILITY, SOLVABILITY AND FIXED-POINT THEOREMS*

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**0. Introduction.** Let $X$, $Y$ be Banach spaces, and $F$ mapping from $X$ into $Y$. In several recent papers [20]-[23], Pochožajev has studied the concept of normal solvability for non-linear mappings. One may sum up some his results as follows. Let $F$ be a Gâteaux-differentiable mapping. Assume that one of the following three conditions is fulfilled:

(1) $Y$ is reflexive, $F(X)$ is weakly closed and

$$\mathrm{Ker}\,[F'(u)]^* = (0) \quad \text{for each } u \in X;$$

(2) $Y$ is uniformly convex, $F(X)$ is closed and

$$\mathrm{Ker}\,[F'(u)]^* = (0) \quad \text{for each } u \in X;$$

(3) $D(F)$ is a linear set in $X$, $Y$ is a uniformly convex Banach space, $F\colon D(F) \to Y$, $F(D(F))$ is closed and

$$\overline{F'(u)\big(D(F)\big)} = Y \quad \text{for each } u \in D(F).$$

Then the image of $F$ is all of $Y$.

Several surjectivity theorems for a Fréchet-differentiable and weakly continuous mapping with weakly closed range and satisfying the condition

$$\|[F'(u)]^* v\| \geqslant k(u)\,\|v\|,$$

where $k(u) > 0$ for each $u \in X$, $v \in Y^*$, have been established by Kačurovskij [13]. Recently, Browder (see [2] and [3]) has considerably sharpened and generalized results of Pochožajev and Kačurovskij by the use of a different method. Assertion (3) of Pochožajev has been extended for an arbitrary Banach space $Y$ by Zabrejko and Krasnoselskij [32], while Daneš [5] has derived a geometrical theorem which can serve as a unification of the initial argument for both papers [2] and [32].

The purpose of this paper is to derive new surjectivity, existence, and fixed-point theorems for non-linear maps and non-linear operator equations. Theorems derived in Section 2 are related to those of Pochožajev, but we do not employ the concept of differentiability of maps. Our method is based on local approximation arguments of a given map by a suitable $p$-positively homogeneous operator and on the minimization properties of certain functionals. In Section 3 new existence theorems concerning solvability of non-linear equations are established. Moreover, the method allows to obtain, among others, new fixed-point theorems for weakly continuous maps and weak local contractions. In comparison with the Banach contraction principle we need not assume that $X$ is complete, $\emptyset \neq M \subset X$, $M$ is closed, and $F$ is a contraction mapping of $M$ into $M$. Let us remark that our hypotheses are different from those of Banach and the authors cited below. Results are related to those of Edelstein [7], Belluce and Kirk [1], Kirk [15], Daneš [6], Reinermann [26] and others.

**1. Terminology and notation.** Let $X$, $Y$ be normed linear spaces, id an identity map of $X$, and $E_1$ a set of real numbers. We use the symbols $\to$ and $\rightharpoonup$ to denote the strong and weak convergence, respectively. Throughout this paper by weak closedness, relative weak compactness and weak compactness we mean respective notions defined by means of sequences. Recall that $p \in X$ is said to be an *internal point* of a subset $W \subset X$ if, for each $u \in X$, there exists an $\varepsilon > 0$ such that

$$|\delta| \leqslant \varepsilon \Rightarrow p + \delta u \in W.$$

The set of all internal points of $W$ is denoted by $\mathrm{Int}_a W$. By $B_\delta(u_0)$ we denote an open ball centered about $u_0$ with the radius $\delta$. For $V \subset X$, $\partial V$ is the boundary of $V$ and $\bar{V}$ denotes its closure in $X$. A mapping $F\colon X \to Y$ is said to be

(1) *$p$-positively homogeneous* if $F(tu) = t^p F(u)$ for each $u \in X$ and $t \geqslant 0$, where $p$ is a positive number;

(2) *weakly continuous at* $u_0 \in X$ if

$$u_n \rightharpoonup u_0 \Rightarrow F(u_n) \rightharpoonup F(u_0);$$

(3) *demicontinuous at* $u_0$ if

$$u_n \to u_0 \Rightarrow F(u_n) \rightharpoonup F(u_0).$$

A functional $f$ is said to be

(a) *weakly lower-semicontinuous at* $u_0$ if $u_n \rightharpoonup u_0$ implies

$$f(u_0) \leqslant \varliminf_{n\to\infty} f(u_n);$$

(b) *quasi-convex on a convex set* if

$$u, v \in M, \lambda \in [0, 1] \Rightarrow f(\lambda\mu + (1-\lambda)v) \leqslant \max[f(u), f(v)].$$

**2. Normal solvability.** The following two lemmas will be useful throughout the paper. The first one is an extension of a classical result, while the second was proved in [17]. Both were applied in [17] in investigations of quasi-convex functionals. We present their proofs here for the sake of completeness.

LEMMA 1. *Let $X$ be a normed linear space, $\emptyset \neq M$ a subset of $X$, and $f$ a real function on $M$. Then $f$ is weakly lower-semicontinuous on $M$ iff $E(c) = \{u \in M\colon f(u) \leqslant c\}$ is weakly closed in $M$ for each $c \in E_1$.*

Proof. Suppose that $f$ is weakly lower-semicontinuous on $M$, $u_n \in E(c)$, $u_n \rightharpoonup u_0$, and $u_0 \in M$. Then

$$f(u_0) \leqslant \varliminf_{n\to\infty} f(u_n) \leqslant c$$

and, therefore, $u_0 \in E(c)$. Conversely, let $(u_n) \in M$ be such that $u_n \rightharpoonup u_0$ and $u_0 \in M$. Then there exists a subsequence $(u_{n_k})$ of $(u_n)$ such that

$$f(u_{n_k}) \to \varliminf_{n\to\infty} f(u_n) = a.$$

Assume that $a < f(u_0)$. Let $\beta$ be a constant such that $a < \beta < f(u_0)$. Then there exists an integer $k_0$ such that $k \geqslant k_0$ implies $f(u_{n_k}) \leqslant \beta$. Hence $u_{n_k} \in E(\beta)$ for $k \geqslant k_0$, and since $u_{n_k} \rightharpoonup u_0$, it follows from our hypothesis that $f(u_0) \leqslant \beta$, a contradiction.

LEMMA 2. *Let $X$ be a normed linear space, $\emptyset \neq M \subseteq X$ a subset of $X$, and $f$ a real weakly lower-semicontinuous functional on $M$. Suppose that the set $E(a) = \{u \in M\colon f(u) \leqslant a\}$ is relatively weakly compact in $M$ for some $a \in E_1$. Then $f$ is bounded from below on $M$. Moreover, if $E(a) \neq \emptyset$, then there exists a point $u_0 \in E(a)$ such that*

$$f(u_0) = \inf\{f(u)\colon u \in M\}.$$

Proof. If $E(a) = \emptyset$, then the first assertion of the lemma is obvious. Suppose that $E(a) \neq \emptyset$ and that $E(a)$ is relatively weakly compact for some $a \in E_1$. If $f$ were not bounded from below on $M$, there would exist a $u_n \in M$ such that $f(u_n) < -n$ for $n = 1, 2, \ldots$ There exists an integer $n_0$ such that $n \geqslant n_0$ implies $f(u_n) \leqslant a$. Hence $u_n \in E(a)$ for each $n \geqslant n_0$. Since the set $E(a)$ is relatively weakly compact in $M$, we infer for a subsequence $(u_{n_k})$ that $u_{n_k} \rightharpoonup p_0$ and $p_0 \in M$. Moreover, by Lemma 1, $E(a)$ is weakly closed in $M$. Therefore, $p_0 \in E(a)$. By the hypothesis,

$$f(p_0) \leqslant \varliminf_{k\to\infty} f(u_{n_k}) = \lim_{k\to\infty} f(u_{n_k}) = -\infty,$$

a contradiction.

Setting $d = \inf\{f(u)\colon u \in M\}$, we have $d \leqslant a$. If $d = a$, then $u \in E(a)$ implies $f(u) = a$ and $f$ attains its lower bound on $E(a)$. If $d < a$, we choose a number $\varepsilon > 0$ such that $d+\varepsilon < a$. There exists a sequence $(v_n) \in M$ such that $f(v_n) \to d$ and, therefore, $f(v_n) \leqslant d+\varepsilon$ for each $n \geqslant n_1$. Hence $n \geqslant n_1 \Rightarrow v_n \in E(a)$. Passing to a subsequence, we infer that $v_{n_k} \rightharpoonup u_0$ and $u_0 \in E(a)$. Then

$$d \leqslant f(u_0) \leqslant \varliminf_{k\to\infty} f(v_{n_k}) = \varliminf_{n\to\infty} f(v_{n_k}) = d$$

and this completes the proof of our lemma.

Let $S$ be a closed set in a Banach space $X$. Then $X$ is said *to have property* (E) if the set of all points $u \in X$ which have a nearest point $s$ in $S$, that is, such that

$$\|u-s\| = \inf\{\|u-v\|: v \in S\},$$

is dense in $X$.

In his remark, Swaminathan [28] has extended the Pochožajev theorem for range spaces having property (E) instead of uniformly convex spaces. It has been proved that the following Banach spaces have property (E):

(a) uniformly convex Banach spaces (Edelstein [8]);

(b) (2R)-Banach spaces (or 2-fully convex Banach spaces of Fan and Glicksberg [9]), i.e. the Banach spaces which satisfy the following condition (H): if $(u_n)$ is such that $\|u_n+u_m\| \to 2$ as $m, n \to \infty$, then $(u_n)$ is a Cauchy sequence;

(c) uniformly smooth Banach spaces satisfying condition (H): if $u_n \rightharpoonup u$ and $\|u_n\| \to \|u\|$, then $u_n \to u$.

The last two assertions have been proved by Wulbert [31]. Let us remark that uniform convexity of Banach space $X$ implies (2R)-property of $X$ and (2R) implies (H). Moreover, each Banach space satisfying one of conditions (a), (b) or (c) is reflexive.

THEOREM 1. *Let $X$ be a normed linear space, and $F$: $X \to X$ a mapping such that, for some $\lambda > 0$,*

(A) *if $u, v \in X$, $u \neq v$, then*

$$\big\|u-v-\lambda\big(F(u)-F(v)\big)\big\| < \|u-v\|.$$

*Let one of the following three conditions be fulfilled:*

(1) *$F$ is weakly continuous,*

$$E(a) = \{u \in X: \|F(u)\| \leqslant a\}$$

*is relatively weakly compact for each $a \geqslant 0$, and $F(0) = 0$;*

(2) *$X$ is reflexive and $F(X)$ is weakly closed;*

(3) *a Banach space $X$ has property* (E) *and $F(X)$ is closed.*

*Then $F$ is surjective.*

Proof. Assume (1) and let $v_0 \in X$, $v_0 \neq 0$, be arbitrary (but fixed). Define a mapping $G\colon X \to X$ by

$$G(u) = u - \lambda F(u) + \lambda v_0, \qquad u \in X,$$

and a functional $f\colon X \to E_1$ by

$$f(u) = \|\lambda(F(u) - v_0)\|, \qquad u \in X.$$

Then $f(u) = \|u - G(u)\|$, $u \in X$, and $f$ is weakly lower-semicontinuous on $X$, because $G$ is weakly continuous. Set

$$A(c) = \{u \in X\colon f(u) \leqslant c\}.$$

Since $0 \in A(\lambda\|v_0\|)$, $A(\|\lambda v_0\|) \neq \emptyset$. If $u \in A(\lambda\|v_0\|)$, then

$$\|\lambda(F(u) - v_0)\| \leqslant \lambda\|v_0\|,$$

and hence $\|F(u)\| \leqslant 2\|v_0\|$. Therefore, $A(\lambda\|v_0\|) \subset E(2\|v_0\|)$. Since $E(2\|v_0\|)$ is relatively weakly compact, also the set $A(\lambda\|v_0\|)$ is. Moreover, $f$ is finite. By Lemma 2, there exists a $u^* \in E(2\|v_0\|)$ such that

$$f(u^*) = \inf\{f(u)\colon u \in X\}.$$

Suppose that $f(u^*) \neq 0$, i.e., $u^* \neq G(u^*)$. According to (A),

$$f(G(u^*)) = \|G(u^*) - G(G(u^*))\| < \|u^* - G(u^*)\| = f(u^*),$$

a contradiction. Hence $u^* = G(u^*)$ and $F(u^*) = v_0$. Since $v_0$ was an arbitrary point, $F(X) = X$.

Assuming (2), for a given $w \in X$ there exists, by Pochožajev's lemma [20], $u^* \in X$ such that

$$f(u^*) = \inf\{\|\lambda(F(u) - w)\|\colon u \in X\}.$$

In the same way as in (1) we infer that $F(u^*) = w$. Assuming (3), we conclude that, for a given point $v \in X$, there exists a sequence $(v_n) \in X$ such that $v_n \to v$ and each $v_n$ has a nearest point in $F(X)$. Hence there exist points $u_n \in X$ such that

$$\|F(u_n^*) - v_n\| = \inf\{\|F(u) - v_n\|\colon u \in X\} \qquad \text{for each } n = 1, 2, \ldots$$

But (A) implies that $F(u_n^*) = v_n$, $n = 1, 2, \ldots$ Hence $v_n \in F(X)$. Since $v_n \to v$ and $F(X)$ is closed, $v \in F(X)$. Therefore, $F(X) = X$.

THEOREM 2. *Let $X$, $Y$ be normed linear spaces, $F\colon X \to Y$ a given mapping, and $D\colon X \to Y$ a suitable $p$-positively homogeneous mapping of $X$ onto $Y$. Suppose that for each point $u \in X$ there exists a set $U(u)$ such that $u \in \operatorname{Int}_a U(u)$, a mapping $G_u\colon X \to Y$, and a constant $a_u \geqslant 0$ such that*

$$\|F(v) - F(u) - G_u(v - u)\| \leqslant a_u \|D(v - u)\|$$

*for each* $v \in U(u)$. *Suppose that there exists a set* $W \subset X$ *with* $0 \in \mathrm{Int}_a W$ *and such that for each* $u \in X$ *there exists an* $\varepsilon_u$ $(0 \leqslant \varepsilon_u < 1 - a_u)$ *satisfying*

$$\|G_u(z) - D(z)\| \leqslant \varepsilon_u \|D(z)\| \quad \textit{for each } z \in W.$$

*Moreover, let one of the following three conditions be fulfilled:*

(a) $F$ *is weakly continuous,* $F(0) = 0$, *and*

$$E(a) = \{u \in X\colon \|F(u)\| \leqslant a\}$$

*is relatively weakly compact for each* $a \geqslant 0$;

(b) $Y$ *reflexive and* $F(X)$ *is weakly closed*;

(c) $Y$ *is complete,* $Y$ *has property* (E), *and* $F(X)$ *is closed.*

*Then* $F(X) = Y$.

**Proof.** Assuming (a), take an arbitrary $v_0 \in Y$, $v_0 \neq 0$. Then $f\colon X \to E_1$, where $f(u) = \|F(u) - v_0\|$ and $u \in X$ is weakly lower-semicontinuous on $X$. The set

$$A(\|v_0\|) = \{u \in X\colon f(u) \leqslant \|v_0\|\}$$

is weakly closed in $X$ and it is a subset of weakly compact set $E(2\|v_0\|)$. Hence $A(\|v_0\|)$ is weakly compact in $X$. Moreover, $A(\|v_0\|) \neq \emptyset$, since $F(0) = 0$. According to Lemma 2, there exists a $u^* \in X$ such that

$$f(u^*) = \inf\{f(u)\colon u \in X\}.$$

Assuming (b) (here and also in (c) we need not suppose that $v_0 \neq 0$), in virtue of Pochožajev's lemma [20] there exists a $u^* \in X$ such that

$$f(u^*) = \inf\{f(u)\colon u \in X\}.$$

Suppose (c); in view of property (E) of $Y$, we conclude that there exist $v_n \in Y$ and $u_n^* \in X$ such that $v_n \to v_0$ and

$$\|F(u_n^*) - v_n\| = \inf\{\|F(u) - v_n\|\colon u \in X\}$$

for each fixed $n = 1, 2, \ldots$

It is sufficient to prove that $f(u^*) = 0$ in cases (a) and (b), and that $F(u_n^*) = v_n$, $n = 1, 2, \ldots$, in case (c). For the proofs of these conditions are almost the same, we shall show only (a).

If $f(u^*) = 0$, then $v_0 = F(u^*)$. Suppose that $f(u^*) \neq 0$. Since $D$ is onto $Y$, there exists a point $\tilde{u} \in X$ such that $D(\tilde{u}) = v_0 - F(u^*)$. By our hypothesis, there exist a set $U(u^*)$ with $u^* \in \mathrm{Int}_a U(u^*)$, a mapping $G_{u^*}\colon X \to Y$, and a constant $a_{u^*} \geqslant 0$ such that

$$\|F(v) - F(u^*) - G_{u^*}(v - u^*)\| \leqslant a_{u^*} \|D(v - u^*)\| \tag{1}$$

holds for each $v \in U(u^*)$. Moreover, there exists a subset $W \subset X$ with $0 \in \mathrm{Int}_a W$ and such that for the point $u^*$ there exists an $\varepsilon_{u^*} \geqslant 0$ satisfying

$$\|G_{u^*}(z) - D(z)\| \leqslant \varepsilon_{u^*} \|D(z)\| \tag{2}$$

for each $z \in W$, where $\varepsilon_{u^*} + a_{u^*} < 1$. Since $u^* \in \mathrm{Int}_a U(u^*)$ and $0 \in \mathrm{Int}_a W$, $u^* + t_0 \tilde{u} \in U(u^*)$ and $t_0 \tilde{u} \in W$ for sufficiently small $t_0 > 0$. Therefore, (1) and (2) imply that

$$(3) \qquad \|F(u^* + t_0 \tilde{u}) - F(u^*) - G_{u^*}(t_0 \tilde{u})\| \leqslant a_{u^*} \|D(t_0 \tilde{u})\| = a_{u^*} t_0^p \|D(\tilde{u})\|,$$

where $a_{u^*} + \varepsilon_{u^*} < 1$, and

$$(4) \qquad \|G_{u^*}(t_0 \tilde{u}) - D(t_0 \tilde{u})\| \leqslant \varepsilon_{u^*} \|D(t_0 \tilde{u})\| = \varepsilon_{u^*} t_0^p \|D(\tilde{u})\|,$$

$$(5) \qquad \begin{aligned} \|F(u^*) - v_0 + D(t_0 \tilde{u})\| &= \|-D(\tilde{u}) + D(t_0 \tilde{u})\| \\ &= \|D(\tilde{u}) - D(t_0 \tilde{u})\| = (1 - t_0^p)\|D(\tilde{u})\|. \end{aligned}$$

Furthermore, by (3), (4) and (5) one gets

$$\begin{aligned} f(u^* + t_0 \tilde{u}) &= \|F(u^* + t_0 \tilde{u}) - v_0\| \\ &\leqslant \|F(u^* + t_0 \tilde{u}) - F(u^*) - G_{u^*}(t_0 \tilde{u})\| + \\ &\qquad + \|F(u^*) - v_0 + D(t_0 \tilde{u})\| + \|G_{u^*}(t_0 \tilde{u}) - D(t_0 \tilde{u})\| \\ &\leqslant [1 - (1 - a_{u^*} - \varepsilon_{u^*}) t_0^p] \|D(\tilde{u})\| < \|D(\tilde{u})\| \\ &= f(u^*), \end{aligned}$$

since $0 \leqslant a_{u^*} + \varepsilon_{u^*} < 1$, $0 < t_0 < 1$, and $D(\tilde{u}) \neq 0$. This contradiction gives $D(\tilde{u}) = 0$. Hence $f(u^*) = 0$ and $F(u^*) = v_0$. Since $v_0 \in Y$, $v_0 \neq 0$, and $F(0) = 0$, we conclude that $F(X) = Y$. This completes the proof.

Remark. The assumption of Theorems 1 and 2 that

$$E(a) = \{u \in X \colon \|F(u)\| \leqslant a\}$$

is relatively weakly compact for each $a \geqslant 0$ is satisfied, for instance, when $X$ is a reflexive Banach space and $F$ is a $p$-positively homogeneous mapping on $X$ such that $\|F(u)\| \geqslant m > 0$ for each $u \in X$, $\|u\| = r$, where $r$ is some positive number.

Setting $G_u = D = G$ for each $u \in X$ in Theorem 2, we obtain the following

COROLLARY 1. *Let $X$, $Y$ be normed linear spaces, $G\colon X \to Y$ a suitable $p$-positively homogeneous mapping of $X$ onto $Y$, and $F\colon X \to Y$ a given map. Suppose that, for each point $u \in X$, there exist an open neighborhood $V(u)$ of $u$ and a constant $a_u$ $(0 \leqslant a_u < 1)$ such that*

$$\|F(v) - F(u) - G(v - u)\| \leqslant a_u \|G(v - u)\|$$

*holds for each $v \in V(u)$. Let one of the three conditions* (a), (b) *or* (c) *of Theorem 2 be fulfilled.*

*Then $F(X) = Y$.*

Definition 1. Let $X$, $Y$ be normed linear spaces, $\emptyset \neq M$ an open subset of $X$, and $A\colon X \to Y$ a mapping. A mapping $\Phi\colon M \to Y$ is said to be an *A-weak local contraction* on $M$ (an *A-local contraction* on $M$) if, for each point $u_0 \in M$, there exist a neighborhood $V(u_0)$ of $u_0$ and a constant $a_{u_0}$ $(0 \leqslant a_{u_0} < 1)$ such that $V(u_0) \subset M$ and

$$\|\Phi(u) - \Phi(u_0)\| \leqslant a_{u_0} \|A(u-u_0)\|$$

for each $u \in V(u_0)$ $(\|\Phi(v)-\Phi(w)\| \leqslant a_{u_0}\|A(v-w)\|$ for each $v, w \in V(u_0))$.

In the case $X = Y$ and $G = \mathrm{id}$, $\Phi$ is called a *weak local contraction* on $M$ (a *local contraction* on $M$).

In comparison with Definition 1 of [16], we need not assume here that $V(u_0)$ is convex and bounded, and that $\overline{V(u_0)} \subset M$. Our concept generalizes the notion of Edelstein [7] and it includes a wider class of mappings.

COROLLARY 2. *Let $X$, $Y$ be normed linear spaces, $A$: $X \to Y$ a linear mapping of $X$ onto $Y$, and $\Phi$: $X \to Y$ an $A$-weak local contraction on $X$. Suppose that one of the three conditions* (a), (b) *or* (c) *of Theorem* 2, *where $F = A+\Phi$, is fulfilled. Moreover, in case* (a), *it is assumed that $A$ is continuous.*

*Then $(A+\Phi)(X) = Y$.*

It is clear that the assumption that $\Phi$ is an $A$-weak local contraction on $X$ can be replaced by the following condition: for each $u_0 \in X$ there exist a neighborhood $V_{u_0}(0)$ of $0$ and a constant $a_{u_0} \in [0, 1)$ such that

$$\|\Phi(u_0+h) - \Phi(u_0)\| \leqslant a_{u_0} \|A(h)\|$$

holds for each $h \in V_{u_0}(0)$.

Setting $\Phi = L$, where $L$: $X \to Y$ is a linear mapping, we obtain the following

COROLLARY 3. *Let $X$, $Y$ be normed linear spaces, and $A$: $X \to Y$, $L$: $X \to Y$ linear maps, where $A$ is onto $Y$. Suppose that $\|L(u)\| \leqslant a\|A(u)\|$ for each $u \in X$ and some $a \in [0, 1)$. Let one of the two following hypotheses hold*:

(1) *$A, L$ are continuous and $\{u \in X: \|(A+L)u\| \leqslant c\}$ is relatively weakly compact for each $c \geqslant 0$;*

(2) *$Y$ is reflexive and $(A+L)(X)$ is closed.*

*Then $(A+L)(X) = Y$.*

This assertion says that under the assumptions of Corollary 3 the equation $(A+L)u = v$ has at least one solution for each $v \in Y$. For results concerning normal solvability and perturbation theory for linear closed operators see Gohberg and Krein [10], Goldberg [11], Przeworska-Rolewicz and Rolewicz [24], and Kato [14].

The simplest conditions (see [29]) which assure the closedness of the range $(A+L)(X)$ are the following: $X$ is complete, $A+L$ is closed and has a continuous inverse $(A+L)^{-1}$. Some necessary and sufficient conditions under which a closed operator has a closed range are given in Chapter IV of [11].

**2. Solvability of non-linear equations and fixed-point theorems.** In this section we derive some general theorems concerning solvability of non-linear equations and, as their corollaries, new fixed-point theorems.

THEOREM 3. *Let $X$, $Y$ be normed linear spaces, $0 \in M \subset X$ an open subset, $F$: $M \to Y$ a given mapping, and $D$: $X \to Y$ a suitable p-positively homogeneous map of $X$ onto $Y$. Suppose that there exists a set $\emptyset \neq N \subset X$ such that $N \subset M$, $N = \mathrm{Int}_a N$, and that*

$$E(c) = \{u \in N:\ \|F(u)\| \leqslant c\}$$

*is relatively weakly compact in $N$ for some $c \geqslant 0$. Assume that for each point $u \in E(c)$ there exist a set $V(u)$ with $u \in \mathrm{Int}_a V(u)$, $V(u) \subset M$, a constant $a_u \geqslant 0$ and a mapping $G_u$: $X \to Y$ such that*

$$(\gamma) \qquad \|F(v)-F(u)-G_u(v-u)\| \leqslant a_u \|D(v-u)\|$$

*holds for each $v \in V(u)$. Moreover, assume that there exists a set $W$ with $0 \in \mathrm{Int}_a W$ and such that for each $u \in E(c)$ there exists an $\varepsilon_u$ $(0 \leqslant \varepsilon_u < 1-a_u)$ satisfying*

$$\|G_u(w)-D(w)\| \leqslant \varepsilon_u \|D(w)\| \quad \textit{for each } w \in W.$$

*If either* (i) *$F$ is weakly continuous on $N$, or* (ii) *$N$ is convex, $F$ is demicontinuous on $N$, and $\psi(u) = \|F(u)\|$ is quasi-convex on $N$, then there exists a $u^* \in N$ such that $F(u^*) = 0$.*

Proof. Assuming (i), we infer that $\psi$ is weakly lower-semicontinuous on $N$. Supposing (ii), we see that $\psi$ is lower-semicontinuous on a convex set $N$. In view of quasi-convexity and lower-semicontinuity of $\psi$, the set $E(d) = \{u \in N:\ \psi(u) \leqslant d\}$ is convex and closed in $N$ for each $d \geqslant 0$. Hence $E(d)$ is weakly closed in $N$. Indeed, if $(u_n) \subset E(d)$, $u_n \rightharpoonup u_0$, and $u_0 \in N$, then, in virtue of Mazur's theorem, there exists a sequence $(v_n)$ of finite convex combinations of $u_n$ such that $v_n \to u_0$. Since $v_n \in E(d)$ and $E(d)$ is closed in $N$, $u_0 \in E(d)$. By Lemma 1, $\psi$ is weakly lower-semicontinuous on $N$. According to Lemma 2, there exists a point $u^* \in E(c) \subset N$ such that

$$\psi(u^*) = \inf\{\psi(u):\ u \in N\}.$$

Since $D$ is onto, there exists a $\tilde{u} \in X$ such that $D(\tilde{u}) = -F(u^*)$. Since $N = \mathrm{Int}_a N$, there exists a number $t_0 > 0$ such that $u^* + t_0 \tilde{u} \in N$. Repeating the argument of the proof of Theorem 1, we conclude that $\psi(u^*) = 0$, i.e., $F(u^*) = 0$. This proves our theorem.

If, in Theorem 1, we set $G_u = D$, $\varepsilon_u = 0$ for each $u \in M$ and $W = \emptyset$, then it is clear that we need not assume that $0 \in M$.

COROLLARY 4. *Let $X$, $Y$ be normed linear spaces, $\emptyset \neq M$ an open subset of $X$, $F\colon M \to Y$ a given mapping, and $G\colon X \to Y$ a suitable p-positively homogeneous map of $X$ onto $Y$. Suppose that there exists a set $\emptyset \neq N \subset X$ such that $N \subset M$, $N = \mathrm{Int}_a N$, and that*

$$E(c) = \{u \in N\colon \|F(u)\| \leqslant c\}$$

*is non-void and relatively weakly compact in $N$ for some $c \geqslant 0$. Assume that for each point $u \in E(c)$ there exists an open neighborhood $V(u)$ of $u$, a constant $a_u$ $(0 \leqslant a_u < 1)$ such that $V(u) \subset M$ and that*

$$\|F(v) - F(u) - G(v-u)\| \leqslant a_u \|G(v-u)\|$$

*holds for each $v \in V(u)$. If either* (i) *$F$ is weakly continuous on $N$, or* (ii) *$N$ is convex, $F$ is demicontinuous on $N$, and $\psi(u) = \|F(u)\|$ is quasi-convex on $N$, then there exists a $u^* \in N$ such that $F(u^*) = 0$.*

COROLLARY 5. *Let $X$, $Y$ be normed linear spaces, $\emptyset \neq M \subset X$ an open subset, $A\colon X \to Y$ a linear continuous map of $X$ onto $Y$, $\Phi\colon M \to Y$ an $A$-weak local contraction on $M$, $\emptyset \neq N \subset M$ a subset in $X$, $N = \mathrm{Int}_a N$. Suppose that $f(u) = \|A(u) + \Phi(u)\|$ is weakly lower-semicontinuous on $N$. If $E(c) = \{u \in N\colon f(u) \leqslant c\}$ is relatively weakly compact in $N$ and non-void for some $c \geqslant 0$, then there is a point $u^* \in N$ such that $A(u^*) + \Phi(u^*) = 0$.*

Setting $X = Y$ and $A = \mathrm{id}$, we obtain the following fixed-point theorem for weak local contraction maps.

COROLLARY 6. *Let $X$ be a normed linear space, $\emptyset \neq M \subset X$ an open subset, $\Phi\colon M \to Y$ a weak local contraction on $M$. Suppose there exists a set $\emptyset \neq N \subset M$, $N = \mathrm{Int}_a N$, such that $\{u \in N\colon \|u - \Phi(u)\| \leqslant c\}$ is non-void and relatively weakly compact in $N$ for some $c \geqslant 0$. If either* (a) *$\Phi$ is weakly continuous on $N$, or* (b) *$N$ is convex, $\Phi$ is demicontinuous on $N$, and $\psi(u) = \|u - \Phi(u)\|$ is quasi-convex on $N$, then there exists a point $u^* \in N$ such that $u^* = \Phi(u^*)$.*

Remark. In comparison with the Banach contraction principle we need not assume in Corollary 5 that $X$ is complete, $M$ is closed, and $\Phi$ is a contraction map of $M$ into $M$. Our conditions are quite different; compare also Edelstein [7], Rakotch [25], Nadler, Jr. [19], and Wong [30], where the Banach contraction principle was extended for single-valued maps and multi-valued local uniform contractions in complete $\varepsilon$-chainable metric spaces.

THEOREM 4. *Let $X$, $Y$ be normed linear spaces, $\emptyset \neq Q$ an open subset of $X$ such that $\bar{Q}$ is weakly compact, and $F\colon \bar{Q} \to Y$ a given mapping. Suppose that for each point $u_0 \in Q$ there exist a $p_{u_0}$-positively homogeneous mapping $G_{u_0}$ of $X$ onto $Y$, a neighborhood $V(u_0)$ of $u_0$, and a constant $a_{u_0} \in [0, 1)$ such that $V(u_0) \subset Q$ and that*

$$(\delta) \qquad \|F(u) - F(u_0) - G_{u_0}(u - u_0)\| \leqslant a_{u_0} \|G_{u_0}(u - u_0)\|$$

*holds for each* $u \in V(u_0)$. *Assume that there is a point* $v_0 \in Q$ *such that* $f(v_0) < f(v)$ *for each* $v \in \partial Q$, *where* $f(v) = \|F(v)\|$, $v \in \bar{Q}$. *If either* (a) $F$ *is weakly continuous on* $\bar{Q}$ *or* (b) $Q$ *is convex*, $F$ *is demicontinuous on* $\bar{Q}$, *and* $f$ *is quasi-convex on* $\bar{Q}$, *then there is a point* $u^* \in Q$ *such that* $F(u^*) = 0$.

Proof. Assuming (a) or (b) we conclude that $f$ is weakly lower-semi-continuous on $\bar{Q}$. Since $\bar{Q}$ is weakly compact, there exists a point $u^* \in \bar{Q}$ such that

$$f(u^*) = \inf\{f(u):\ u \in \bar{Q}\}.$$

Since $f(v_0) < f(v)$ for each $v \in \partial Q$ and $v_0 \in Q$, we infer that $u^* \in Q$. Using the arguments similar to those of the proof of Theorem 2 (we put here $\varepsilon_u = 0$ for each $u \in Q$), we obtain $f(u^*) = 0$, i.e., $F(u^*) = 0$.

Now, similarly as before, one can deduce several consequences of Theorem 4. Here we derive only two simple assertions.

COROLLARY 6. *Let* $X$, $Y$ *be normed linear spaces*, $X$ *reflexive*, $\Phi$: $\overline{B_R(0)} \to Y$ *a weakly continuous mapping, and* $A$: $X \to Y$ *a linear continuous operator of* $X$ *onto* $Y$. *If* $\Phi$ *is an* $A$*-weak local contraction on* $B_R(0)$ *and*

$$\|\Phi(0)\| < \|A(u) + \Phi(u)\| \quad \text{for each} \quad u \in \partial B_R(0),$$

*then* $A(u^*) + \Phi(u^*) = 0$.

PROPOSITION 1. *Let* $X$, $Y$ *be normed linear spaces*, $X$ *reflexive*, $\emptyset \neq C \subset X$ *a weakly closed set*, $M \supset C$ *an open subset of* $X$, $\Phi$: $M \to Y$ *a weakly continuous mapping, and* $A$: $X \to Y$ *a linear continuous operator of* $X$ *onto* $Y$. *If* $\Phi$ *is an* $A$*-weak local contraction on* $M$ *and* $\|A(u) + \Phi(u)\| \to +\infty$ *for each* $u \in C$, $\|u\| \to +\infty$, *then there is a point* $u^* \in C$ *such that* $A(u^*) + \Phi(u^*) = 0$.

Setting in Corollary 6 and Proposition 1 $X = Y$, $A = \mathrm{id}$, $A = T - \mathrm{id}$, respectively, where $T$: $X \to X$ is a linear continuous operator of $X$ into $X$ and $T - \mathrm{id}$ is into, we obtain new fixed-point theorems for the mapping $\Phi$ and for the sum of operators $T$, $\Phi$, respectively.

Let us remark that in Theorems 3 and 4 and in their consequences the mappings $G_u$, $D$ and $G$ need not be defined on the whole space $X$.

Recall that theorems of Section 2 and their consequences give sufficient conditions under which a point $u^*$, at which $f(u) = \|F(u)\|$ takes its minimum value, is simultaneously a solution of the equation $F(u) = 0$. Since there are many approximate methods (see, for instance, [12]) for localization and finding points at which non-linear functionals take its minimum value, these computational methods provide us also with approximate solution of the equation $F(u) = 0$.

Furthermore, it is obvious that assumptions ($\gamma$) and ($\delta$) (or that $\Phi$ is an $A$-weak local contraction) need not be satisfied on $M$ or $Q$ (or on $B_R(0)$), but only in some neighborhood of the point $u^*$. Setting in Corollary 6 $X = Y$ and $A = -\mathrm{id}$, we obtain the following assertion:

*Let $X$ be a reflexive Banach space, and $\Phi$: $\overline{B_R(0)} \to X$ a weakly continuous mapping such that*

$$\|\Phi(0)\| < \|u - \Phi(u)\| \quad \text{for each } u \in \partial B_R(0).$$

*If $\|\Phi(u^* + h) - \Phi(u^*)\| \leqslant a\|h\|$ for some $a \in [0, 1)$ and all $h \in V(0)$, where $u^* \in B_R(0)$ is such that*

$$f(u^*) = \inf\{\|u - \Phi(u)\|: u \in B_R(0)\},$$

*and a neighborhood $V(0)$ of 0 satisfies $u^* + V(0) \subset B_R(0)$, then $u^* = \Phi(u^*)$.*

The last hypothesis can be replaced by the following one: $\Phi$ is Fréchet-differentiable at $u^*$ and $\|\Phi'(u^*)\| = a < 1$.

PROPOSITION 2. *Let $X$ be a normed linear space, $\emptyset \neq M$ a subset of $X$, and $F$: $M \to M$ a mapping such that $\|F(u) - F(v)\| < \|u - v\|$ holds for each $u, v \in M$. If either* (a) *$X$ is reflexive and $(\mathrm{id} - F)(M)$ is weakly closed, or* (b) *$(\mathrm{id} - F)(M)$ is weakly compact, then there is a unique point $u^* \in M$ such that $F(u^*) = u^*$.*

Proof. Assume (a). We shall follow here the argument of Pochožajev [20]. Set

$$d = \inf\{\|u - F(u)\|: u \in M\}.$$

Then there is a sequence $(u_n) \subset M$ such that $\|u_n - F(u_n)\| \to d$. Since $X$ is reflexive and $\big(u_n - F(u_n)\big)$ is bounded, passing to a subsequence we infer that $u_{n_k} - F(u_{n_k}) \rightharpoonup w$ and $w \in (\mathrm{id} - F)(M)$. Hence there exists a $u^* \in M$ such that $u^* - F(u^*) = w$. It is clear that $\|u^* - F(u^*)\| = d$. If $u^* \neq F(u^*)$, then

$$\big\|F(u^*) - F\big(F(u^*)\big)\big\| < \|u^* - F(u^*)\| = d,$$

a contradiction. Uniqueness of $u^*$ is obvious.

Assuming (b), there exists a sequence $v_n \in M$ such that

$$\|v_n - F(v_n)\| \to d = \inf\{\|F(u) - u\|: u \in M\}.$$

In view of the weak compactness of $(\mathrm{id} - F)(M)$, there is a subsequence $\big(u_{n_k} - F(u_{n_k})\big)$ such that

$$u_{n_k} - F(u_{n_k}) \rightharpoonup w_0 \in (\mathrm{id} - F)(M).$$

Now we proceed similarly as above.

PROPOSITION 3. *Let $X$ be a normed linear space, $M \neq \emptyset$ an open subset of $X$, and $F$: $M \to X$ a weak local contraction on $M$. If either* (a) *$X$ is reflexive and $(\mathrm{id} - F)(N)$ is weakly closed for some subset $N$, $\emptyset \neq N \subset M$, $N = \mathrm{Int}_a N$, or* (b) *$X$ is a Banach space with property* (E) *and $(\mathrm{id} - F)(W)$ is closed for some set $\emptyset \neq W \subset M$, $W = \mathrm{Int}_a W$, then there exists a point $u^* \in M$ such that $u^* = F(u^*)$.*

Remark. In comparison with the Browder result [4] we do not assume in Propositions 2 and 3 that $M$ is a convex, closed and bounded subset of $X$, but we put stronger conditions on the mapping $F$ and the space $X$. A preliminary report about a part of these results is contained in [18].

Added in proof. We pointed out that the concept of the $A$-weak local contraction (Definition 1) generalizes the notion of Edelstein [7]. Let us also remark that our concept is more general than that of Kirk's generalized contraction. Compare W. A. Kirk, *On nonlinear mappings of strongly semicontractive type,* Journal of Mathematical Analysis and Applications 27 (1969), p. 409 - 412; *Mappings of generalized contractive type,* ibidem 32 (1970), p. 567 - 572; *Fixed point theorems for nonlinear nonexpansive and generalized contraction mappings,* Pacific Journal of Mathematics 38 (1971), p. 89 - 94.

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*Reçu par la Rédaction le 24. 7. 1972*