COLLOQUIUM MATHEMATICUM

VOL. LX/LXI DÉDIÉ À M. ANTONI ZYGMUND 1990

ONE-SIDED CONDITIONS FOR QUASILINEAR JACOBI DIFFERENTIAL EQUATIONS

BY

VICTOR L. SHAPIRO (RIVERSIDE, CALIFORNIA)

1. Introduction. Let $\Omega = (-1,1)$ and

$$(1.0)\qquad \rho(x) = (1-x)^{\alpha}(1+x)^{\beta} \quad \text{and} \quad p(x) = (1-x)^{\alpha+1}(1+x)^{\beta+1}$$

where $-1 < \alpha < 0$ and $-1 < \beta < 0$. With $Du = du/dx$, we call

$$(1.1)\qquad -D[p(x)A(x,u,Du)] = \rho(x)F(x,u), \quad x \in \Omega$$

a *quasilinear Jacobi differential equation.*

In order to deal with the quasilinear differential equation (1.1), we introduce the two pre-Hilbert spaces:

$$(1.2)\qquad C^0_\rho = \left\{u \in C^0(\Omega) : \int_\Omega u^2\rho < \infty\right\}$$

with $\langle u,v\rangle_\rho = \int_\Omega uv\rho$;

$$(1.3)\qquad C^1_{p,\rho} = \left\{u \in C^0(\overline{\Omega}) \cap C^1(\Omega) : \int_\Omega [|Du|^2p + u^2\rho] < \infty\right\}$$

with $\langle u,v\rangle_{p,\rho} = \int_\Omega [DuDvp + uv\rho]$ where $\overline{\Omega} = [-1,1]$.

L^2_ρ will designate the Hilbert space one gets by completing C^0_ρ using the method of Cauchy sequences with respect to the norm $\|u\|_\rho = \langle u,u\rangle^{1/2}_\rho$, and $H^1_{p,\rho}$ will designate the Hilbert space that one gets by completing $C^1_{p,\rho}$ with respect to the norm $\|u\|_{p,\rho} = \langle u,u\rangle^{1/2}_{p,\rho}$.

For the A appearing in (1.1), we shall make the following assumptions:

(A-1) $A(x,t,\xi) : \Omega\times\mathbf{R}\times\mathbf{R} \to \mathbf{R}$ and satisfies the Carathéodory conditions (i.e., $A(x,t,\xi)$ is measurable in Ω for every fixed $(t,\xi) \in \mathbf{R}\times\mathbf{R}$ and is continuous in (t,ξ) for a.e. fixed $x \in \Omega$).

(A-2) $\exists h \geq 0$ and $c_1 \geq 0$ with $h \in L^2_p$ such that

$$|A(x,t,\xi)| \leq h(x) + c_1(t^2+\xi^2)^{1/2} \quad \text{for a.e. } x \in \Omega.$$

(A-3) $\exists$ a positive constant c_0 and nonnegative function $Z \in L^1_p$ (i.e., $\int_\Omega |Z| p < \infty$) such that

$$A(x,t,\xi)\xi \geq c_0|\xi|^2 - Z(x)$$

for a.e. $x \in \Omega$ and $\forall(t,\xi) \in \mathbf{R} \times \mathbf{R}$.

(A-4) $[A(x,t,\xi) - A(x,t,\xi')](\xi - \xi') > 0$ for a.e. $x \in \Omega$, $\forall t \in \mathbf{R}$, and $\forall \xi, \xi' \in \mathbf{R}$ with $\xi \neq \xi'$.

For $F(x,t)$ in (1.1), we shall suppose $F(x,t) = f(x) - g(x,t)$ where $f \in L^1_\rho$ and $g(x,t)$ meets the following conditions:

(g-1) $g(x,t)$ satisfies the usual Carathéodory condition;
(g-2) for each $r > 0$, $\exists \zeta_r \in L^1_\rho$ such that

$$|g(x,t)| \leq \zeta_r(x) \quad \text{for } |t| \leq r \text{ and a.e. } x \in \Omega .$$

We observe, in particular, that if $g(x,t) \in C^0([-1,1] \times \mathbf{R})$, then g meets (g-1) and (g-2). With A meeting (A-1)–(A-4), g meeting (g-1), (g-2) and $f \in L^1_\rho$, we shall say $u \in H^1_{p,\rho}$ is a *weak solution* of the equation

$$-D[pA(x,u,Du)] + g(x,u)\rho = f(x)\rho$$

if the following holds:

$$\int_\Omega pA(x,u,Du)Dv + \int_\Omega g(x,u)v\rho = \int_\Omega f(x)v\rho \quad \forall v \in H^1_{p,\rho}. \tag{1.4}$$

With p and ρ as defined in (1.0) and $-1 < \alpha, \beta < 0$, we shall show in our second lemma that $H^1_{p,\rho}$ is continuously imbedded in $L^\infty(\Omega)$ (actually in $C^0(\overline{\Omega})$ where $\overline{\Omega} = [-1,1]$). Hence if $u, v \in H^1_{p,\rho}$, it follows from (A-2) and (g-2) that each of the integrals in (1.4) is well defined.

The general one-sided result concerning (1.4) that we shall establish in this paper is the following:

THEOREM 1. *Let $\Omega = (-1,1)$ and let p and ρ be given by* (1.0) *with $-1 < \alpha, \beta < 0$. Suppose that A satisfies* (A-1)–(A-4), *g satisfies* (g-1) *and* (g-2), *and that $f \in L^1_\rho$. Suppose furthermore that*

$$tg(x,t) \geq 0 \quad \text{for } t \in \mathbf{R} \text{ and a.e. } x \in \Omega , \tag{1.5}$$

$$\int_\Omega f\rho = 0 . \tag{1.6}$$

Then $\exists u \in H^1_{p,\rho}$ which is a weak solution of the equation

$$-D[pA(x,u,Du)] + g(x,u)\rho = f(x)\rho ,$$

that is, (1.4) *holds $\forall v \in H^1_{p,\rho}$.*

We observe that g in the above theorem is quite general except for the one-sided condition (1.5). In particular, $g(x,t) = te^{t^4} + t\sin^2 x$ satisfies (g-1), (g-2), and (1.5).

As a corollary to the above theorem, we have the following quasilinear result which is both necessary and sufficient.

THEOREM 2. *Let Ω, p and ρ be as in Theorem 1. Suppose that $f \in L^1_\rho$. Then a necessary and sufficient condition that $\exists u \in H^1_{p,\rho}$ such that u is a weak solution of*

$$-D[pA(x,u,Du)] = f(x)\rho \tag{1.7}$$

is that

$$\int_\Omega f\rho = 0. \tag{1.8}$$

It is clear that by taking $g(x,t) \equiv 0$ the sufficiency condition of Theorem 2 is an immediate corollary to Theorem 1.

To establish the necessary condition in Theorem 2, suppose $u \in H^1_{p,\rho}$ is a weak solution of (1.7). Take $v = 1$ in (1.4) and the proof follows.

For an extension of Theorem 1 above, see Remark 2 in §4.

2. Fundamental lemmas. The first lemma we prove is the following:

LEMMA 1. *Let $v \in H^1_{p,\rho}$ and set $\Gamma_p = \int_\Omega p^{-1}$. Suppose*

$$\int_\Omega v\rho = 0. \tag{2.1}$$

Then

$$|v(x)| \le \Gamma_p \|Dv\|_p \quad \text{for a.e. } x \in \Omega. \tag{2.2}$$

From the definition of $H^1_{p,\rho}$, to establish the above lemma, it is sufficient to show that (2.2) holds for $v \in C^1_{p,\rho}$. To do this, we see from (2.1) that for $x \in (-1,1)$, $v(x) = b_\rho^{-1} \int_\Omega [v(x) - v(t)]\rho(t)\,dt$ where $b_\rho = \langle 1,1\rangle_\rho$. Since $v \in C^1(\Omega)$,

$$v(x) - v(t) = \int_t^x Dv(s)\,ds \quad \text{for } x,t \in \Omega. \tag{2.3}$$

The observation that $p^{-1} \in L^1(\Omega)$ joined with (2.3) gives in a standard manner (2.2) where $\|Dv\|_p^2 = \int_\Omega |Dv|^2 p$.

Remark 1. *If $v \in H^1_{p,\rho}$, then $\exists w \in C^0(\overline{\Omega})$ such that $v(x) = w(x)$ a.e. in Ω.*

Remark 1 follows easily from Lemma 1.

LEMMA 2. *Let* $v \in H^1_{p,\rho}$. *Then* $|v(x)| \leq \Gamma_p \|Dv\|_p + b_\rho^{-1/2}\|v\|_\rho$ *for a.e.* $x \in \Omega$ *where* $b_\rho = \langle 1, 1\rangle_\rho$.

Setting $z(x) = v(x) - b_\rho^{-1}\langle v, 1\rangle \rho$, we see that Lemma 2 follows from Lemma 1.

LEMMA 3. $H^1_{p,\rho}$ *is compactly imbedded in* L^2_ρ.

The proof of Lemma 3 is an immediate consequence of Remark 1, Lemma 2, the well-known Ascoli–Arzelà theorem, and the following fact: if $\{v_n\}_{n=1}^\infty$ is a sequence in $C^1_{p,\rho}$ with $\int_\Omega |Dv_n|^2 p \leq K \ \forall n$, then the sequence is uniformly equicontinuous on $\overline{\Omega}$.

At this point, we introduce the Jacobi polynomials $P_n^{\alpha,\beta}(x)$. In particular, $P_n^{\alpha,\beta}(x)$ are polynomials of degree n that satisfy the equation

$$-D[pDP_n^{\alpha,\beta}] = n(n+\alpha+\beta+1)\rho P_n^{\alpha,\beta} \tag{2.4}$$

and are usually normalized so that $P_n^{\alpha,\beta}(1) = \binom{n+\alpha}{n}$. Jacobi polynomials are dealt with in the literature in various places, particularly in [3], [10], and [4, Chap. 8].

As a consequence of (2.4), we see that

$$\langle P_n^{\alpha,\beta}, v\rangle_{p,\rho} - \langle P_n^{\alpha,\beta}, v\rangle_\rho = n(n+\alpha+\beta+1)\langle P_n^{\alpha,\beta}, v\rangle_\rho \quad \forall v \in H^1_{p,\rho}. \tag{2.5}$$

We shall set

$$\varphi_n(x) = P_n^{\alpha,\beta}(x)/\|P_n^{\alpha,\beta}\|_\rho. \tag{2.6}$$

Therefore $\|\varphi_n\|_\rho = 1$ and we take it as well known (see [10] or [3, pp. 31–32 and 39–40]) that

$$\{\varphi_n\}_{n=0}^\infty \text{ is a complete orthonormal system in } L^2_\rho \text{ with } \varphi_0 \text{ a positive constant.} \tag{2.7}$$

Also, we set

$$\lambda_n = n(n+\alpha+\beta+1)+1 \quad \text{for } n = 0, 1, 2, \ldots, \tag{2.8}$$

and observe from (2.5), (2.6), and (2.8) that

$$\langle \varphi_n, v\rangle_{p,\rho} = \lambda_n \langle \varphi_n, v\rangle_\rho \quad \forall v \in H^1_{p,\rho}. \tag{2.9}$$

From (2.8) we see that $\lambda_n < \lambda_{n+1}$ and $\lambda_0 = 1$. Consequently, it follows from (2.7) and (2.9) that

$$\{\varphi_n/\lambda_n^{1/2}\}_{n=0}^\infty \text{ is a CONS in } H^1_{p,\rho}. \tag{2.10}$$

For convenience of notation, we introduce the two-form

$$\mathcal{Q}(u, v) = \int_\Omega pA(x, u, Du)Dv \tag{2.11}$$

defined for $u, v \in H^1_{p,\rho}$.

LEMMA 4. *Let n be a positive integer, and let S_n be the subspace of $H^1_{p,\rho}$ spanned by $\{\varphi_k\}_{k=0}^n$. Assume the hypotheses of Theorem 1. Then $\exists u_n \in S_n$ such that*

$$Q(u_n, v) + \int_\Omega g(x, u_n) v\rho + n^{-1}\langle u_n, v\rangle_\rho = \int_\Omega f v\rho \quad \forall v \in S_n . \tag{2.12}$$

To establish the above lemma, we let $\gamma = (\gamma_0, \ldots, \gamma_n) \in \mathbf{R}^{n+1}$ and using the summation convention set

$$\begin{aligned} G_k(\gamma) = Q(\gamma_j\varphi_j, \varphi_k) &+ \int_\Omega g(x, \gamma_j\varphi_j)\varphi_k\rho \\ &+ n^{-1}\langle \gamma_j\varphi_j, \varphi_k\rangle_\rho - \int_\Omega f\varphi_k\rho , \end{aligned} \tag{2.13}$$

for $k=0,\ldots,n$. Then with $G(\gamma)=(G_0(\gamma),\ldots,G_n(\gamma))$ we obtain from (2.7)

$$\begin{aligned} G(\gamma)\cdot\gamma = Q(\gamma_j\varphi_j, \gamma_k\varphi_k) &+ \int_\Omega g(x, \gamma_j\varphi_j)(\gamma_k\varphi_k)\rho \\ &+ n^{-1}|\gamma|^2 - \int_\Omega f(\gamma_k\varphi_k)\rho . \end{aligned} \tag{2.14}$$

From (A-3), (2.16), (1.5), and (1.6), we see from (2.14) that

$$G(\gamma)\cdot\gamma \geq -\int_\Omega Zp + n^{-1}|\gamma|^2 - \hat{f}(k)\gamma_k , \tag{2.15}$$

where $Z \in L^1_p$ and

$$\hat{f}(k) = \int_\Omega f\varphi_k\rho . \tag{2.16}$$

It is clear from (2.15) that

$$\exists r_1 > 0 \quad \text{such that} \quad G(\gamma)\cdot\gamma > 0 \quad \text{for } |\gamma| \geq r_1 . \tag{2.17}$$

It is also clear that $G_k \in C^0(\mathbf{R}^{n+1})$ for $k = 0,\ldots,n$. Hence it follows from (2.17) and [7, p. 18] that $\exists\gamma^\# = (\gamma_0^\#,\ldots,\gamma_n^\#)$ such that $G_k(\gamma^\#) = 0$ for $k = 0,\ldots,n$. We set $u_n = \gamma_j^\#\varphi_j$ and find from (2.13) that

$$Q(u_n, \varphi_k) + \int_\Omega g(x, u_n)\varphi_k\rho + n^{-1}\langle u_n, \varphi_k\rangle_\rho = \int_\Omega f\varphi_k\rho \tag{2.18}$$

for $k = 0,\ldots,n$. From (2.11) we see that Q is linear in its second variable, i.e., as a function of v. Since every $v \in S_n$ is a finite linear combination of $\{\varphi_k\}_{k=0}^n$, (2.12) is an immediate consequence of (2.18), and the proof of Lemma 4 is complete.

3. Proof of Theorem 1. To prove Theorem 1, we invoke Lemma 4 and obtain a sequence of functions $\{u_n\}_{n=1}^{\infty}$ such that

$$u_n \in S_n \text{ and } u_n \text{ satisfies (2.12) for } n = 1,2,\ldots \tag{3.1}$$

We claim

$$\exists K_1 > 0 \quad \text{such that} \quad \|u_n\|_{p,\rho} \le K_1 \quad \forall n. \tag{3.2}$$

Suppose (3.2) is false. Then there exists a subsequence (which, for ease of notation, we take to be the full sequence) such that

$$\lim_{n\to\infty} \|u_n\|_{p,\rho} = \infty\,. \tag{3.3}$$

To show that (3.3) leads to a contradiction, we set

$$w_n = \widehat{u}_n(0)\varphi_0 \quad \text{and} \quad z_n = u_n - w_n\,, \tag{3.4}$$

where $\widehat{u}_n(0) = \langle u_n, \phi_0\rangle_\rho$.

We take $v = u_n$ in (2.12) and deduce from (1.5), (1.6), (3.4) and the fact that $\varphi_0 =$ a constant that

$$Q(u_n, u_n) \le \int_\Omega f z_n \rho\,. \tag{3.6}$$

Now from (2.17) and (3.4), we see that

$$Q(u_n, u_n) = \int_\Omega pA(x, u_n, Dz_n)Dz_n\,.$$

Hence, it follows from (3.6) and (A-3) that

$$c_0 \int_\Omega |Dz_n|^2 p \le \int_\Omega f z_n \rho + \int_\Omega Zp \quad \forall n\,, \tag{3.7}$$

where c_0 is a positive constant and Z is a nonnegative function in L_p^1. It follows from (3.4) that $\langle z_n, \varphi_0\rangle_\rho = 0$. So we conclude from Lemma 1 and (3.7) that

$$c_0 \int_\Omega |Dz_n|^2 p \le \Gamma_p \int_\Omega |f|\rho \Big\{ \int_\Omega |Dz_n|^2 p \Big\}^{1/2} + \int_\Omega Zp \quad \forall n\,.$$

Since c_0 is a positive constant, it follows from this last inequality that

$$\exists K_2 > 0 \quad \text{such that} \quad \int_\Omega |Dz_n|^2 p \le K_2 \quad \forall n\,. \tag{3.8}$$

Since $u_n \in S_n$ it follows that $u_n \in C^0(\overline{\Omega})$. Consequently, we see from (3.4) that $z_n \in C^0(\overline{\Omega})$. Therefore using (3.8) in conjunction with Lemma 1, we obtain

$$\exists K_3 > 0 \quad \text{such that} \quad |z_n(x)| \le K_3 \quad \forall x \in \Omega \text{ and } \forall n\,. \tag{3.9}$$

Now from (3.4), we see that

$$\langle u_n, u_n\rangle_\rho = |\widehat{u}_n(0)|^2 + \langle z_n, z_n\rangle_\rho . \tag{3.10}$$

Also, since $\varphi_0 =$ a constant, we find from (3.4) that

$$\int_\Omega |Du_n|^2 p = \int_\Omega |Dz_n|^2 p, \tag{3.11}$$

and furthermore from (1.2) and (3.9) that

$$\langle z_n, z_n\rangle_\rho \leq K_3^2 \int_\Omega \rho \quad \forall n . \tag{3.12}$$

Next, from (1.2), (1.3), and (3.3), we see that

$$\lim_{n\to\infty} \int_\Omega |Du_n|^2 p + \langle u_n, u_n\rangle_\rho = \infty .$$

But then it follows from this fact in conjunction with (3.8) and (3.10)–(3.12) that

$$\lim_{n\to\infty} |\widehat{u}_n(0)|^2 = \infty . \tag{3.13}$$

As a consequence of this last fact, we have the existence of a subsequence of $\{\widehat{u}(0)\}_{n=1}^\infty$ which tends to ∞ or $-\infty$. We shall assume the existence of a subsequence (which for ease of notation we take to be the full sequence) which goes to ∞ and arrive at a contradiction. A similar argument will apply in case the subsequence goes to $-\infty$. Hence, we assume that (3.13) implies that

$$\lim_{n\to\infty} \widehat{u}_n(0) = \infty , \tag{3.14}$$

and will show that this leads to a contradiction.

To do this, first of all we observe from (1.5) that there exists a set $E \subset \Omega$ of Lebesgue measure zero such that

$$g(x,t) \geq 0 \quad \forall x \in \Omega - E \text{ and } \forall t \geq 0 . \tag{3.15}$$

Next we observe from (3.4) that

$$u_n(x) = \widehat{u}_n(0)\varphi_0 + z_n(x) \quad \forall n , \tag{3.16}$$

where φ_0 is a positive constant. From (3.14), it follows that $\exists n_0$ such that

$$\widehat{u}_n(0)\varphi_0 \geq K_3 + 1 \qquad \text{for } n \geq n_0 . \tag{3.17}$$

Consequently, it follows from (3.9), (3.16), and (3.17) that

$$u_n(x) \geq 1 \qquad \text{for } x \in \Omega \text{ and } n \geq n_0 . \tag{3.18}$$

From (3.15) and this last fact, we conclude that

$$g(x, u_n(x)) \geq 0 \qquad \text{for } x \in \Omega - E \text{ and } n \geq n_0 , \tag{3.19}$$

where $E \subset \Omega$ is a set of Lebesgue measure zero. Next using (3.1) and selecting $v = \varphi_0$ in (2.18), we see that

$$\mathcal{Q}(u_n, \varphi_0) + \int_\Omega g(x, u_n)\varphi_0\rho + n^{-1}\langle u_n, \varphi_0\rangle_\rho = \int_\Omega f\varphi_0\rho. \tag{3.20}$$

Now from the fact that φ_0 is a positive constant, we see from (2.17) that $\mathcal{Q}(u_n, \varphi_0) = 0$ and from (1.6) that $\int_\Omega f\varphi_0\rho = 0$. Therefore, we conclude from (3.5) and (3.20) that

$$\int_\Omega g(x, u_n)\varphi_0\rho + n^{-1}\widehat{u}_n(0) = 0 \quad \forall n. \tag{3.21}$$

But from (3.19), the integral in (3.21) is nonnegative for $n \geq n_0$. Consequently, $n^{-1}\widehat{u}_n(0) \leq 0$ for $n \geq n_0$, and we obtain $\widehat{u}_n(0) \leq 0$ for $n \geq n_0$. This inequality is a direct contradiction of (3.14). We conclude that (3.3) is false and (3.2) is indeed true.

Next we invoke Lemma 3 and obtain the existence of a subsequence of $\{u_n\}_{n=1}^\infty$ (which, for ease of notation, we take to be the full sequence) and a function

$$u \in H^1_{p,\rho} \tag{3.22}$$

such that the following facts prevail:

$$\lim_{n\to\infty} \|u_n - u\|_\rho = 0; \tag{3.23}$$

$$\lim_{n\to\infty} u_n(x) = u(x), \quad \text{a.e. in } \Omega; \tag{3.24}$$

$$\lim_{n\to\infty} \int_\Omega Du_n wp = \int_\Omega Duwp \quad \forall w \in L^2_p. \tag{3.25}$$

We propose to show also there exists a subsequence $\{u_{n_j}\}_{j=1}^\infty$ such that

$$\lim_{j\to\infty} Du_{n_j}(x) = Du(x), \quad \text{a.e. in } \Omega. \tag{3.26}$$

To establish (3.26), it is sufficient to establish the following two facts:

(1) $\exists$ a subsequence $\{u_{n_j}\}_{j=1}^\infty$ such that

$$\begin{aligned}\lim_{j\to\infty} &[A(x, u_{n_j}, Du_{n_j}) - A(x, u_{n_j}, Du)] \\ &\times [Du_{n_j}(x) - Du(x)] = 0 \quad \text{for a.e. } x \in \Omega;\end{aligned} \tag{3.27}$$

(2) with $\{u_{n_j}\}_{j=1}^\infty$ designating the same subsequence as in (3.27),

$$\{|Du_{n_j}(x)|\}_{j=1}^\infty \text{ is pointwise bounded for a.e. } x \in \Omega. \tag{3.28}$$

The fact that (3.27) and (3.28) together imply (3.26) via (A-4) and (3.24) is a standard technique in the theory of quasilinear differential equations (see [5] or [6]), and we leave the details to the reader. For an explicit situation similar to the above situation, we refer the reader to [9, (2.34)–(2.36)].

Hence to establish (3.26), it remains to show that (3.27) and (3.28) hold. To show that (3.27) holds, we show separately that

$$\lim_{n\to\infty} \int_\Omega A(x,u_n,Du)[Du_n - Du]p = 0 \tag{3.29}$$

and

$$\lim_{n\to\infty} \int_\Omega A(x,u_n,Du_n)[Du_n - Du]p = 0\,. \tag{3.30}$$

It is clear from (A-4) and [8, p. 70] that (3.29) and (3.30) together imply (3.28). We leave the details to the reader. (For an analogous situation in the literature, see [9, (2.40) and (2.41)].)

(3.29) follows immediately from the following two easily established facts:

$$\text{(i)}\qquad \lim_{n\to\infty} \int_\Omega A(x,u,Du)[Du_n - Du]p = 0\,,$$

$$\text{(ii)}\qquad \lim_{n\to\infty} \int_\Omega [A(x,u_n,Du) - A(x,u,Du)][Du_n - Du]p = 0\,.$$

It remains to establish (3.30). From (2.11) we see that (3.30) is equivalent to showing

$$\lim_{n\to\infty} \mathcal{Q}(u_n, u_n - u) = 0\,. \tag{3.31}$$

To establish (3.31), define

$$P_n u = \sum_{k=0}^{n} \widehat{u}(k)\varphi_k\,, \tag{3.32}$$

where $\widehat{u}(k) = \langle u, \varphi_k\rangle_\rho$. Then $P_n u \in S_n$ as defined in Lemma 4. Also, it is easy to see from (2.7)–(2.10) that

$$\lim_{n\to\infty} \|u - P_n u\|_{p,\rho} = 0\,. \tag{3.33}$$

It consequently follows from (A-2) and (3.2) that

$$\lim_{n\to\infty} \mathcal{Q}(u_n, P_n u - u) = 0\,. \tag{3.34}$$

Hence (3.31) will follow once we show that

$$\lim_{n\to\infty} \mathcal{Q}(u_n, u_n - P_n u) = 0\,. \tag{3.35}$$

To establish (3.35), we observe from (3.32) that $P_n u \in S_n$. Therefore it follows from (3.1) and (2.12) that

$$Q(u_n, u_n - P_n u) = -\langle u_n, u_n - P_n u\rangle_\rho n^{-1} + \int_\Omega f[u_n - P_n u]\rho - \int_\Omega g(x, u_n)[u_n - P_n u]\rho. \tag{3.36}$$

From (3.2) and Lemma 2, we see that $\exists K_5$ such that $|u_n(x)| \leq K_5\ \forall n$ and $\forall x \in \Omega$. Hence from (g-2) we see $\exists \zeta \in L^1_\rho$ such that $|g(x, u_n)| \leq \zeta(x)\ \forall n$ and a.e. in Ω. Also from (3.24) and (3.33), we see that $\lim_{n\to\infty}(u_n - P_n u) = 0$ a.e. in Ω. We therefore conclude that the last term on the right-hand side of (3.36) goes to zero as $n \to \infty$. A similar situation prevails for the first two terms. Hence (3.35), and consequently (3.27), is established.

(3.28) follows from standard technique in the theory of quasilinear elliptic equations. We leave the details to the reader. (See [9, (2.48)] for a similar situation.) This completes the proof of (3.26).

To complete the proof of Theorem 1, let $v \in \bigcup_{n=1}^\infty S_n$, say $v \in S_{n_0}$. Then it follows from (3.1) and (2.12) that for $n_j \geq n_0$,

$$Q(u_{n_j}, v) + \int_\Omega g(x, u_{n_j})v\rho + n_j^{-1}\langle u_{n_j}, v\rangle_\rho = \int_\Omega fv\rho. \tag{3.37}$$

Now from (A-1), (3.24), and (3.26), we have

$$\lim_{j\to\infty} |A(x, u_{n_j}, Du_{n_j}) - A(x, u, Du)|\,|Dv|p = 0, \quad \text{a.e. in } \Omega. \tag{3.38}$$

Also, it follows from (3.2), (3.22), and (A-2) that

$$\exists K_6 > 0 \quad \text{such that} \quad \int_\Omega |A(x, u_n, Du_n) - A(x, u, Du)|^2 p \leq K_6 \quad \forall n.$$

This fact in conjunction with the fact that $Dv \in L^2_p$ and Schwarz's inequality gives that $\{|A(x, u_n, Du_n) - A(x, u, Du)|\,|v|p\}_{n=1}^\infty$ is an absolutely equiintegrable sequence. This fact in turn, when combined with (3.38) and Egorov's theorem [8, p. 75], gives that

$$\lim_{n\to\infty} \int_\Omega A(x, u_{n_j}, Du_{n_j})Dvp = \int_\Omega A(x, u, Du)Dvp. \tag{3.39}$$

The same idea used to establish (3.39) shows that

$$\lim_{j\to\infty} \int_\Omega g(x, u_{n_j})v\rho = \int_\Omega g(x, u)v\rho. \tag{3.40}$$

Since $|\langle u_{n_j}, v\rangle_\rho| \leq \|u_{n_j}\|_\rho \|v\|_\rho$, we furthermore deduce from (3.2) that $\lim_{j\to\infty} n_j^{-1}\langle u_{n_j}, v\rangle_\rho = 0$. This last fact in conjunction with (3.37), (3.39), and (3.40) gives that

$$(3.41)\qquad Q(u,v)+\int_{\Omega} g(x,u)v\rho=\int_{\Omega} fv\rho\quad \forall v\in\bigcup_{n=1}^{\infty} S_n\,.$$

Finally, let $v \in H^1_{p,\rho}$. Then we define $P_n v$ as in (3.32) and observe as in (3.33) that $\lim_{n\to\infty}\|v-P_nv\|_{p,\rho}=0$. Since $A(x,u,Du)\in L^2_p$, it follows from (2.11) and this last fact that

$$(3.42)\qquad \lim_{n\to\infty} Q(u,P_nv)=Q(u,v)\,.$$

In a similar manner, we see that

$$(3.43)(a)\qquad \lim_{n\to\infty}\int_{\Omega} g(x,u)P_nv\rho=\int_{\Omega} g(x,u)v\rho\,,$$

$$(3.43)(b)\qquad \lim_{n\to\infty}\int_{\Omega} fP_nv\rho=\int_{\Omega} fv\rho\,.$$

Since $P_nv\in S_n$, we see that (3.41) holds with v replaced by P_nv. But then (3.42) and (3.43)(a) and (b) show that (3.41) holds for all $v\in H^1_{p,\rho}$ and the proof of Theorem 1 is complete.

4. Concluding remarks. Theorem 1 is capable of being generalized. With this in mind, we now leave $\rho(x)$ and $p(x)$ have the following properties in this section:

$$(4.1)\qquad \rho\in C^0(\Omega)\,,\qquad \rho(x)>0\quad \forall x\in\Omega\,,\qquad \int_{\Omega}\rho<\infty\,,$$

$$(4.2)\qquad p\in C^0(\overline{\Omega})\,,\qquad p(x)>0\quad \forall x\in\Omega\,,\qquad \int_{\Omega}p^{-1}<\infty\,,$$

$$(4.3)\qquad \exists K>0\quad \text{such that}\quad p(x)\le K\rho(x)\quad \forall x\in\Omega\,.$$

We then define C^0_ρ and $C^1_{p,\rho}$ as before and L^q_ρ, L^q_p, and $H^1_{p,\rho}$ as before for $1\le q<\infty$. We also take (A-1)–(A-4) and (g-1), (g-2) as before and (1.4) continues to stand for a weak solution of $-D[p(A(x,u,Du))]+g(x,u)\rho=f(x)\rho$.

Remark 2. *Theorem* 1 *continues to hold if* ρ *and* p *meet* (4.1)–(4.3).

To establish Remark 2, we observe that Lemma 1 continues to hold with $\Gamma_p=[\int_\Omega p^{-1}]^{1/2}$. The same holds for Lemma 2. Also we see that Lemma 3 still remains true and also Remark 1 continues to hold. The question is: do we have the analogue of (2.7)–(2.10)? The answer is yes. To see this, we define $\mathcal{L}(u,v)=\int_\Omega pDuDv+\rho uv$ for $u,v\in H^1_{p,\rho}$ and define the Raleigh quotient $J(u)$ to be

$$J(u) = \mathcal{L}(u,u)/\langle u,u\rangle_\rho, \quad u \not\equiv 0, \ u \in H^1_{p,\rho}.$$

Then proceeding exactly as in [2, pp. 213–214], we use the compactness of the imbedding $H^1_{p,\rho}$ in L^2_ρ given by Lemma 3 to obtain a sequence $\{\varphi_n\}_{n=0}^\infty$ with the properties (2.7)–(2.10) where $\lambda_0 = 1$ and $\{\lambda_n\}_{n=0}^\infty$ instead of being given explicitly by (2.8) is now a nondecreasing sequence of real numbers with $\lim_{n\to\infty} \lambda_n = \infty$. We also note as before that φ_0 is a positive constant and λ_0 is a simple eigenvalue.

We observe that Lemma 4 continues to hold with S_n defined with respect to this new orthonormal sequence. Also the proof of Theorem 1 goes through in a similar manner since all the lemmas hold. We leave the interested reader to check the details.

A typical $A(x,t,\xi)$ that will work for the theorems presented in this paper is $A(x,t,\xi) = a(x)b(t)[\xi + G(|\xi|)\xi]$ where $a \in C^0(\overline{\Omega})$, $b \in L^\infty(\mathbf{R}) \cap C^0(\mathbf{R})$, and G is a bounded continuous nondecreasing function on $[0,\infty)$ with $G(0) \geq 0$. Also we suppose $\exists c_2 > 0$ such that $a(x) \geq c_2$ $\forall x \in \overline{\Omega}$ and $b(t) \geq c_2$ $\forall t \in \mathbf{R}$.

In closing, we point out that as a special case our theorems do cover the familiar Chebyshev polynomials [1, pp. 567–568] and [3, p. 40 and pp. 46–49]. As a sequel to this paper, we shall present further, but different results involving special functions and quasilinear partial differential equations.

REFERENCES

[1] W. E. Boyce and R. C. Di Prima, *Elementary Differential Equations and Boundary Value Problems*, 3rd ed., Wiley, New York 1977.

[2] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, 2nd ed., Springer, Berlin 1983.

[3] H. Hochstadt, *The Functions of Mathematical Physics*, Wiley-Interscience, New York 1971.

[4] D. Jackson, *Fourier Series and Orthogonal Polynomials*, Carus Math. Monographs 6, Math. Assoc. Amer., 1948.

[5] J. Leray and J.-L. Lions, *Quelques résultats de Višik sur les problèmes elliptiques non linéaires par les méthodes de Minty-Browder*, Bull. Soc. Math. France 93 (1965), 97–107.

[6] J. Nečas, *Introduction to the Theory of Nonlinear Elliptic Equations*, Teubner-Texte zur Math. 52, Teubner, Leipzig 1983.

[7] L. Nirenberg, *Topics In Nonlinear Functional Analysis*, Lecture Notes, Courant Institute, New York 1974.

[8] W. Rudin, *Real and Complex Analysis*, 2nd ed., McGraw-Hill, New York 1974.

[9] V. L. Shapiro, *Resonance and the second BVP*, Trans. Amer. Math. Soc., to appear.

[10] G. Szegö, *Orthogonal Polynomials*, 4th ed., Amer. Math. Soc. Colloq. Publ. 23, Providence 1978.

DEPARTMENT OF MATHEMATICS & COMPUTER SCIENCE
UNIVERSITY OF CALIFORNIA
RIVERSIDE, CALIFORNIA 92521-0135, U.S.A.

Reçu par la Rédaction le 6.3.1990;
en version modifiée le 20.6.1990