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*VARIETIES WITH ISOMORPHIC FREE ALGEBRAS**

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Introduction. It has been shown by Goetz and Ryll-Nardzewski [2] that if F_n, $n = 1, 2, \ldots$, is the free algebra on n generators in some variety V and for some $p \neq q$, $F_p \cong F_q$, then the natural numbers r for which $F_p \cong F_r$ form an arithmetic sequence. Thus F_p has a free generating set, or basis, of r elements for each such number r. Conversely, Świerczkowski [6] proved that any arithmetic sequence of natural numbers is the set of all powers of bases of some free algebra. More specifically, let $V_{(m,n)}$ be the variety of algebras with n-ary operations $\theta_1, \theta_2, \ldots, \theta_m$ and m-ary operations $\pi_1, \pi_2, \ldots, \pi_n$ satisfying identities

$$\theta_i \pi_1 x_1 \ldots x_m \pi_2 x_1 \ldots x_m \ldots \pi_n x_1 \ldots x_m = x_i \quad (i = 1, 2, \ldots, m)$$

and

$$\pi_j \theta_1 y_1 \ldots y_n \theta_2 y_1 \ldots y_n \ldots \theta_m y_1 \ldots y_n = y_j \quad (j = 1, 2, \ldots, n),$$

and let $\{k+sd \mid s = 0, 1, 2, \ldots\}$ be any arithmetic sequence of natural numbers. Świerczkowski showed that

THEOREM *The numbers in the sequence* $\{k+sd \mid s = 0, 1, 2, \ldots\}$ *are precisely the powers of bases of the free algebra on* k *generators in any variety* $V_{(n,n+d)}$, *provided* $n \leqslant k < n+d$.

In this paper, we will obtain this theorem as a consequence of some general results on the structure of the subalgebras of the free $V_{(m,n)}$-algebras. We solve the word problem for free $V_{(m,n)}$-algebras and obtain a normal form for the elements of these algebras by using the technique developed by Evans [1] for the word problems for loops, quasigroups, and other multiplicative systems. (A general form of this technique has been described by Knuth [3].) We also give a counterexample for problem (P 526) proposed by Marczewski [5].

1. The word problem for free algebras. Let F be the free algebra on $\{a_1, a_2, \ldots\}$ in $V_{(m,n)}$ (where no restriction is meant on the cardinality of the generating set). The elements of F are equivalence classes of well-

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formed $V_{(m,n)}$ words on symbols $\{a_1, a_2, \ldots\}$, where equivalence is defined in terms of the identities for $V_{(m,n)}$. Every word will be considered to be a *subword* of itself. If $u_1, u_2, \ldots, u_n$ (or $v_1, v_2, \ldots, v_m$) are words, then they are the major subwords of $\theta_i u_1 u_2 \ldots u_n$ (or $\pi_j v_1 v_2 \ldots v_m$). It will often be convenient to denote a word $\theta_i w_1 w_2 \ldots w_n$ by $\theta_i \{w_j\}_{j=1}^n$, and a word $\pi_j w_1 w_2 \ldots w_m$ by $\pi_j \{w_i\}_{i=1}^m$, when $w_1, w_2, \ldots$ are themselves complex expressions. A *reduction* of a word w is a word formed by replacing a subword $\theta_i\{\pi_j u_1 u_2 \ldots u_m\}_{j=1}^n$ (or $\pi_j\{\theta_i v_1 v_2 \ldots v_n\}_{j=1}^m$) by u_i (or v_j). A word is *reduced* if no reduction is possible. We define the *rank* of a word w, denoted by $R(w)$, as follows: The rank of a generator is one, and the rank of any other word is one more than the maximum of the ranks of its major subwords. We also define the *length* of a word to be the total number of generators and operation symbols occurring in it.

LEMMA 1. *If $w = u_0, u_1, \ldots, u_k$ and $w = v_0, v_1, \ldots, v_l$ are word sequences such that u_i and v_j are reductions of u_{i-1} and v_{j-1}, respectively, and u_k and v_l are reduced, then u_k and v_l are the same word.*

Proof. We show that u_1 and v_1 can always be reduced to the same word. The proof then follows by induction on the length of w. If the reductions $w \to u_1$ and $w \to v_1$ occur in different subwords of w, then u_1 and v_1 clearly have a common reduction. If they occur in the same proper subword, the lemma follows by induction. Suppose, then, that the reduction $w \to u_1$ involves all of w. Then w has the form $\pi_j\{\theta_i t_1 t_2 \ldots t_n\}_{i=1}^m$ (or analogously, $\theta_i\{\pi_i t_1 t_2 \ldots t_m\}_{i=1}^n$), where u_1 is t_j. If $u_1 = v_1$, the lemma follows by induction. Otherwise the reduction $w \to v_1$ occurs in some subword $\theta_g t_1 t_2 \ldots t_n$ of w. There are three possibilities here:

(i) If $w \to v_1$ occurs in some $t_p, p \neq j$, then this reduction may be copied in the other occurrences of t_p, and the resulting word may be reduced to u_1.

(ii) If $w \to v_1$ occurs in t_j, replacing it by t_j', then this reduction may be copied in each other occurrence of t_j, and the result reduced to t_j', which is also a reduction of u_1.

(iii) If $w \to v_1$ involves all of $\theta_g t_1 t_2 \ldots t_n$, then there are words $r_1, r_2, \ldots, r_m$ such that, for each p, $t_p = \pi_p r_1 r_2 \ldots r_m$, and v_1 is obtained by replacing $\theta_g t_1 t_2 \ldots t_n$ by r_g. Then v_1 can be reduced by replacing $\theta_h t_1 t_2 \ldots t_n$ by r_h for each $h \neq g$. But this reduces v_1 to $\pi_j r_1 r_2 \ldots r_m = t_j = u_1$.

It follows from Lemma 1 that every word has a unique reduced form. The following theorem provides a solution to the word problem for F:

THEOREM 1. *Words u and v represent the same element of the free algebra F if and only if their reduced forms are the same.*

Proof. If u and v can be reduced to the same word, they certainly represent the same element of F. If u and v represent the same element of F, there is a sequence $u = w_0, w_1, \ldots, w_k = v$ such that for each pair $\{w_i, w_{i+1}\}$, one is a reduction of the other. The theorem follows by induction on k, using Lemma 1.

2. Subalgebras of free algebras. In view of Theorem 1, we will identify elements of F with the reduced words representing them. By the *rank of an element* we mean the rank of this reduced word. A subset S of F will be called *irredundant* if no element g of S is in the subalgebra generated by the elements of S with rank less than that of g. Otherwise S will be called *redundant*. A subset of F will be called *condensed* if it contains no subset of the form $\{\theta_i u_1 u_2 \ldots u_n\}_{i=1}^{m}$ or $\{\pi_j v_1 v_2 \ldots v_m\}_{j=1}^{n}$. A subset of F is *independent in the sense of Marczewski* [4] if it freely generates a subalgebra of F. In this section we will show that the property of being condensed and irredundant is sufficient to assure that a subset of F is independent. We will then show that any subalgebra of F contains such a generating set and is therefore free.

Let $X_{(m,n)}$ be the free $V_{(m,n)}$-algebra on a countably infinite set $\{x_1, x_2, x_3, \ldots\}$ of free generators. As in F, the elements of $X_{(m,n)}$ are represented as reduced words on the generators. If $u(x_1, x_2, \ldots, x_r)$ and $v(x_1, x_2, \ldots, x_r)$ are in $X_{(m,n)}$, then $u(x_1, x_2, \ldots, x_r) = v(x_1, x_2, \ldots, x_r)$ if and only if $u(h_1, h_2, \ldots, h_r) = v(h_1, h_2, \ldots, h_r)$ for any subset $\{h_1, h_2, \ldots, h_r\}$ of an algebra of $V_{(m,n)}$.

LEMMA 2. *If $u(x_1, x_2, \ldots, x_r)$ and $v(x_1, x_2, \ldots, x_r)$ are reduced words in $X_{(m,n)}$, $\{g_1, g_2, \ldots, g_r\}$ is a condensed and irredundant subset of r distinct elements of F, $u(g_1, g_2, \ldots, g_r)$ and $v(g_1, g_2, \ldots, g_r)$ are reduced in F, and $u(g_1, g_2, \ldots, g_r) = v(g_1, g_2, \ldots, g_r)$, then $u(x_1, x_2, \ldots, x_r) = v(x_1, x_2, \ldots, x_r)$.*

Proof. Suppose $R(u) \geqslant R(v)$. We proceed by induction on $R(u)$. If $R(u) = 1$, then $u(g_1, g_2, \ldots, g_r) = g_i = g_j = v(g_1, g_2, \ldots, g_r)$. Hence, $i = j$ and $u(x_1, x_2, \ldots, x_r) = x_i = v(x_1, x_2, \ldots, x_r)$. Now assume the lemma is true for pairs of words of rank less than $R(u)$, where $R(u) > 1$. We consider two cases.

Case 1. $R(v) = 1$. Let $u(x_1, x_2, \ldots, x_r) = \beta u_1(x_1 \ldots x_r) u_2(x_1 \ldots x_r) \ldots$, where β denotes some one of the $m+n$ operations. If $v(x_1, x_2, \ldots, x_r) = x_i$, then $u(g_1, g_2, \ldots, g_r) = \beta u_1(g_1, g_2, \ldots, g_r) u_2(g_1, g_2, \ldots, g_r) \ldots = g_i = v(g_1, g_2, \ldots, g_r)$. Since both sides are reduced in F, g_i is expressed in terms of the elements of smaller rank, contradicting the irredundance of $\{g_1, g_2, \ldots, g_r\}$.

Case 2. $R(v) > 1$. Let $u(x_1, x_2, \ldots, x_r) = \beta u_1(x_1 \ldots x_r) u_2(x_1 \ldots x_r) \ldots$ and $v(x_1, x_2, \ldots, x_r) = \beta' v_1(x_1 \ldots x_r) v_2(x_1 \ldots x_r) \ldots$ Then

$$\beta u_1(g_1 \ldots g_r) u_2(g_1 \ldots g_r) \ldots = \beta' v_1(g_1 \ldots g_r) v_2(g_1 \ldots g_r) \ldots$$

Since these are both reduced in F, by Theorem 1 they must be identical, i.e., $\beta = \beta'$ and $u_i(g_1 \ldots g_r) = v_i(g_1 \ldots g_r)$. By induction,

$u_i(x_1 \ldots x_r) = v_i(x_1 \ldots x_r)$ for each i, so that $u(x_1, x_2, \ldots, x_r) = v(x_1, x_2, \ldots, x_r)$.

LEMMA 3. *If $\{g_1, g_2, \ldots, g_r\}$ is a condensed and irredundant subset of F and $u(x_1, x_2, \ldots, x_r)$ is a reduced word in $X_{(m,n)}$, then $u(g_1, g_2, \ldots, g_r)$ is reduced in F.*

Proof. The proof is by induction on the rank of $u(x_1, x_2, \ldots, x_r)$. If $R(u) = 1$, then $u(x_1, x_2, \ldots, x_r) = x_i$, so $u(g_1, g_2, \ldots, g_r) = g_i$ which is reduced. Assume the lemma is true for words of rank less than $R(u)$, where $R(u) > 1$. Let

$$u(x_1, x_2, \ldots, x_r) = \pi_j\{u_i(x_1, x_2, \ldots, x_r)\}_{i=1}^m$$

(or analogously, $u(x_1, x_2, \ldots, x_r) = \theta_i\{u_j(x_1, x_2, \ldots, x_r)\}_{j=1}^n$). Then each subword $u_k(x_1, x_2, \ldots, x_r)$ of the reduced word $u(x_1, x_2, \ldots, x_r)$ must be reduced, so, by induction, each $u_k(g_1, g_2, \ldots, g_r)$ is reduced.

Now suppose that $u(g_1, g_2, \ldots, g_r)$ can be reduced. We will show that $u(x_1, x_2, \ldots, x_r)$ can similarly be reduced, contradicting the hypothesis that $u(x_1, x_2, \ldots, x_r)$ is itself a reduced word. Since each major subword of $u(g_1, g_2, \ldots, g_r)$ is reduced, the reduction can not occur within a proper subword of $u(g_1, g_2, \ldots, g_r)$. Hence, each $u_k(g_1, g_2, \ldots, g_r)$ must be the word $\theta_k w_1(a_1, a_2, \ldots) w_2(a_1, a_2, \ldots) \ldots w_n(a_1, a_2, \ldots)$ for some fixed set $\{w_1(a_1, a_2, \ldots), \ldots, w_n(a_1, a_2, \ldots)\}$ of reduced words in F. Then $w_j(a_1, a_2, \ldots)$ is a reduction of $u(g_1, g_2, \ldots, g_r)$ in F:

$$u(g_1, g_2, \ldots, g_r) = \pi_j\{\theta_k w_1(a_1, a_2, \ldots) \ldots w_n(a_1, a_2, \ldots)\}_{k=1}^m = w_j(a_1, a_2, \ldots)$$

We must first show that no word $u_k(x_1, x_2, \ldots, x_r)$ has rank 1. Suppose, then, that $R(u_k) = 1$ for some u_k. Then $u_k(g_1, g_2, \ldots, g_r)$ is a member of $\{g_1, g_2, \ldots, g_r\}$. Since $u_k(g_1, g_2, \ldots, g_r)$ is reduced, and $\{g_1, g_2, \ldots, g_r\}$ is irredundant, we must conclude that some $w_p(a_1, a_2, \ldots)$ is not in the subalgebra of F generated by $\{g_1, g_2, \ldots, g_r\}$. But since $w_p(a_1, a_2, \ldots)$ is a major subword of each $u_i(g_1, g_2, \ldots, g_r)$, it follows that

$$u_i(g_1, g_2, \ldots, g_r) = \theta_i w_1(a_1, a_2, \ldots) \ldots w_n(a_1, a_2, \ldots)$$

is a member of $\{g_1, g_2, \ldots, g_r\}$ for $i = 1, 2, \ldots, m$. This contradicts the assumption that $\{g_1, g_2, \ldots, g_r\}$ is condensed.

Since, for each i, $u_i(g_1, g_2, \ldots, g_r)$ is reduced and $u_i(x_1, x_2, \ldots, x_r)$ is a word of rank greater than 1, $u_i(x_1, x_2, \ldots, x_r)$ must have the form

$$u_i(x_1, x_2, \ldots x_r) = \theta_i\{u_i^k(x_1, x_2, \ldots, x_r)\}_{k=1}^n.$$

Then for any h and i such that $1 \leqslant h,\ i \leqslant m$,

$$u_h^k(g_1, g_2, \ldots, g_r) = w_k(a_1, a_2, \ldots) = u_i^k(g_1, g_2, \ldots, g_r).$$

By Lemma 2, we conclude that

$$u_h^k(x_1, x_2, \ldots, x_r) = u_i^k(x_1, x_2, \ldots, x_r)[= u^k(x_1, x_2, \ldots, x_r)],$$

so that

$$u(x_1, x_2, \ldots, x_r) = \pi_j\{\theta_i u^i(x_1, x_2, \ldots, x_r)\ldots u^n(x_1, x_2, \ldots, x_r)\}_{i=1}^m,$$

which is not reduced.

LEMMA 4. *Every condensed, irredundant subset S of F is free.*

Proof. Suppose $u(g_1, g_2, \ldots, g_r) = v(g_1, g_2, \ldots, g_r)$, where $\{g_1, g_2, \ldots \ldots, g_r\} \subseteq S$, $R(g_i) \leqslant R(g_{i+1})$, and the words $u(x_1, x_2, \ldots, x_r)$ and $v(x_1, x_2, \ldots, x_r)$ are reduced. By Lemma 3, $u(g_1, g_2, \ldots, g_r)$ and $v(g_1, g_2, \ldots \ldots, g_r)$ must be reduced, as words in F. Then by Lemma 2, $u(x_1, x_2, \ldots \ldots, x_r) = v(x_1, x_2, \ldots, x_r)$.

LEMMA 5. *If S generates a subalgebra $\langle S\rangle$ of F, then there is a condensed, irredundant subset S' of F that generates the same subalgebra.*

Proof. Define a sequence $\{S_i\}$ of subsets of F as follows: $S_1 = S$. If $u \in S_i$, then $u \in S_{i+1}$. If $\{\pi_k u_1 u_2 \ldots u_m\}_{k=1}^n \subseteq S_i$ and $[\{\theta_k v_1 v_2 \ldots v_n\}_{k=1}^m \subseteq S_i]$, then $\{u_1, u_2, \ldots, u_m\} \subseteq S_{i+1}$ and $[\{v_1, v_2, \ldots, v_n\} \subseteq S_{i+1}]$.

Note that $\langle S\rangle = \langle S_1\rangle = \langle S_2\rangle = \ldots = \langle S_i\rangle = \ldots$

Let $\bar{S} = \bigcup_{i=1}^{\infty} S_i$. Then $\langle S\rangle = \langle \bar{S}\rangle$. Let T be the union of all subsets of $\bar{S}$ of the form $\{\pi_k u_1 \ldots u_m\}_{k=1}^n$ or $\{\theta_k v_1 \ldots v_n\}_{k=1}^m$. The set $\bar{S}-T$ is clearly condensed.

We now claim that $\langle \bar{S}-T\rangle = \langle S\rangle$. Indeed, since $\bar{S}-T \subseteq \bar{S}$, we have $\langle \bar{S}-T\rangle \subseteq \langle \bar{S}\rangle = \langle S\rangle$. To show that $\langle \bar{S}-T\rangle = \langle S\rangle$, we must show that $S \subseteq \langle \bar{S}-T\rangle$. If $s \in S$ and $s \notin T$, then $s \in \bar{S}-T$. If $s \in T$, by induction on $R(s)$ we show that $s \in \langle \bar{S}-T\rangle$: If $R(s) = 1$, $s \notin T$. Assume that elements of T of rank less than $R(s)$ are in $\langle \bar{S}-T\rangle$. Let $s = \pi_j v_1 v_2 \ldots v_m$ (or analogously, $s = \theta_i u_1 u_2 \ldots u_n$). Since $s \in T$, we have $\{\pi_k v_1 v_2 \ldots v_m\}_{k=1}^n \subseteq \bar{S}$, so $\{\pi_k v_1 v_2 \ldots v_m\}_{k=1}^n \subseteq S_p$ for some p. Then $\{v_1, v_2, \ldots, v_m\} \subseteq S_{p+1} \subseteq \bar{S}$. Now, if $v_i \in T$, by induction, $v_i \in \langle \bar{S}-T\rangle$. If $v_i \notin T$, then $v_i \in \bar{S}-T$. Hence $s \in \langle \bar{S}-T\rangle$. We now construct an irredundant subset of $\bar{S}-T$ that generates the same subalgebra. Let r be the least rank of any element of $\bar{S}-T$ and, for each $t \geqslant r$, let H_t be all elements of $\bar{S}-T$ of rank less than or equal to t. Define

$$S' = \{h \in \bar{S}-T \mid h \notin \langle H_{R(h)-1}\rangle\} \cup H_r.$$

Then S' is clearly condensed and irredundant. To show that $\langle S'\rangle = \langle \bar{S}-T\rangle (= \langle S\rangle)$, we write $\bar{S}-T$ as the ascending union

$$\bar{S}-T = \bigcup_{k=r}^{\infty} H_k$$

and show by induction on k that $H_k \subseteq \langle S'\rangle$. $H_r \subseteq \langle S'\rangle$ since $H_r \subseteq S'$. Suppose $H_k \subseteq \langle S'\rangle$ and let $x \in H_{k+1}$. If $x \in \langle H_k\rangle$, then $x \in \langle S'\rangle$. If $x \notin \langle H_k\rangle$, then x has rank $k+1$ so that $x \in S' \subseteq \langle S'\rangle$.

As an immediate consequence of Lemma 4 and Lemma 5 we have

THEOREM 2. *Every subalgebra of a free $V_{(m,n)}$-algebra is free.*

3. Isomorphic free algebras. Let F_t denote the free $V_{(m,n)}$-algebra on generators $\{a_1, a_2, \ldots, a_t\}$.

LEMMA 6. *The only condensed free generating set for F_t is $\{a_1, a_2, \ldots, a_t\}$.*

Proof. Let S be a condensed free generating set for F_t. Then S is also irredundant. Let $\{g_1, g_2, \ldots, g_r\} \subseteq S$ and $w_1, w_2, \ldots, w_t$ be reduced words such that $w_i(g_1, g_2, \ldots, g_r) = a_i$. By Lemma 3, $w_i(g_1, g_2, \ldots, g_r)$ is reduced so that $w_i(x_1, x_2, \ldots, x_r) = x_k$, where $g_k = a_i$. Hence $\{a_1, a_2, \ldots, \ldots, a_t\} \subseteq S$. Since S is irredundant, it contains nothing else.

LEMMA 7. *If the subset $\{u_k\}_{k=1}^{n}$ [or $\{v_k\}_{k=1}^{m}$] of F is independent, the subset $\{\theta_i u_1 u_2 \ldots u_n\}_{i=1}^{m}$ [or $\{\pi_j v_1 v_2 \ldots v_n\}_{j=1}^{n}$] is also independent and generates the same subalgebra.*

Proof. Let G be an algebra in $V_{(m,n)}$ and let $\gamma : \theta_i u_1 u_2 \ldots u_n \to g_i$ be a mapping into G. We must show that γ can be extended to a homomorphism. Define $a : F \to G$ as the extension of $u_j \to \pi_j g_1 g_2 \ldots g_m$ to a homomorphism. Then

$$(\theta_i u_1 u_2 \ldots u_n) a = \theta_i \{\pi_j g_1 g_2 \ldots g_m\}_{i=1}^{n} = g_i,$$

so that a is an extension of γ to a homomorphism. Since $\pi_j \{\theta_i u_1 u_2 \ldots u_n\}_{i=1}^{m} = u_j$, the subsets generate the same subalgebra.

We now use these results to give an alternative proof of Świerczkowski's theorem:

THEOREM 3. *Suppose d, n, p, and q are positive integers such that $p > q$. Then, in the variety $V_{(n,n+d)}$, $F_p \cong F_q$ if and only if $p, q \geqslant n$ and $p \equiv q \bmod (d)$.*

Proof. Suppose first that $p, q \geqslant n$ and $p = q + rd$. Let F_q be freely generated by $\{a_1, a_2, \ldots, a_q\}$.

Since $q \geqslant n$, applying Lemma 7 we can replace any n elements $\{u_1, u_2, \ldots, u_n\} \subseteq \{a_1, a_2, \ldots, a_q\}$ by the $n+d$ elements $\{\theta_i u_1 u_2 \ldots u_n\}_{i=1}^{n+d}$ and obtain a free set of $q+d$ generators for F_q. If this is done r times, we obtain a free set of $q+rd = p$ generators for F_q. Hence, $F_q \cong F_p$.

If $F_p \cong F_q$, let $\{a_1, a_2, \ldots, a_q\}$ and $\{b_1, b_2, \ldots, b_p\}$ be free generating sets for F_q. Consider a sequence $B_1, B_2, B_3, \ldots$ of subsets of F_q such that $B_1 = \{b_1, b_2, \ldots, b_p\}$ and B_i is constructed from B_{i-1} by replacing a subset $\{\theta_i u_1 u_2 \ldots u_n\}_{i=1}^{n+d}$ by $\{u_1, u_2, \ldots, u_n\}$ or $\{\pi_j v_1 v_2 \ldots v_m\}_{j=1}^{n}$ by $\{v_1, v_2, \ldots, v_{n+d}\}$. By Lemma 7, each B_i freely generates F_q. Now, if B is any finite subset of F_q, define $|B|$ to be the sum of the lengths of its elements. Then the sequence $|B_1|, |B_2|, |B_3|, \ldots$ is decreasing and must therefore be finite. Let B_s be the last term of $B_1, B_2, \ldots$ Then B_s is a condensed generating set for F_q; by Lemma 6, $B_s = \{a_1, a_2, \ldots, a_q\}$. Since B_i

is constructed from B_{i-1} by adding or subtracting d elements, $p \equiv q \mod (d)$. Since $p \neq q, r \geqslant 2$. Constructing B_2 from B_1 and B_r from B_{r-1} required exchanging n elements for $n+d$ elements, or $n+d$ for n, so that both p and q must be greater than or equal to n.

It follows from Theorem 3 that, given the arithmetic sequence of natural numbers $\{k+sd \mid s = 0, 1, 2, \dots\}$, the powers of bases of the free algebra F_k in any variety $V_{n,n+d)}$, where $n \leqslant k < n+d$ are exactly the numbers in this sequence.

Goetz and Ryll-Nardzewski [2] proved that if an algebra has a basis of one element and a basis of $n > 1$ elements, then for every positive integer k there exists a minimal set of k self-dependent generators of the algebra (x is *self-dependent* if $\{x\}$ is not independent). Marczewski [5] raised the following question (P 526): Can an analogous theorem be deduced from the weakened assumption that the algebra has bases of different numbers of elements? Athough a negative answer has already been established (1), we point out that the algebras examined in this paper present a further counterexample. In fact, let F_n be the free algebra on n generators in $V_{(m,n)}$, where neither m nor n is 1. If x is in F_n, then $\{x\}$ is irredundant; since m and n are different from 1, it is also condensed. By Lemma 4, $\{x\}$ is independent. Hence, F_n contains no self-dependent elements. Moreover, it is clear that every finite subset of F_n contains an irredundant subset generating the same subalgebra. A subset of F_n containing less than n elements must be condensed, so, in view of Lemma 6, no subset of less than n elements can generate F_n.

(1) Cf. Colloquium Mathematicum 17 (1967), P 526, R1, p. 367.

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