COLLOQUIUM MATHEMATICUM

VOL. LIX 1990 FASC. 1

SIMPLE PLANE IMAGES OF THE SIERPIŃSKI TRIANGULAR CURVE ARE NOWHERE DENSE

BY

W. DĘBSKI AND J. MIODUSZEWSKI (KATOWICE)

The image of a closed interval of reals under a continuous map into the plane is nowhere dense if the map assumes each value at most twice. This property of the interval, proved by Hahn [2] and Mazurkiewicz [9], is shared also, as was shown by Hurewicz [3], by curves X such that

(α) each non-degenerate subcontinuum of X has non-empty interior.

It is shared also by arbitrary dendrites; the latter is a result of Sieklucki announced by Lelek [7] (1). However, the Sierpiński universal plane curve does not have this property, as its holes can be sewn in such a way that the resulting space is a plane square (*sewing* means that each point is sewn with at most one other); see [5]. So the property does not follow from one-dimensionality only.

Following Borsuk and Molski [1], a continuous map assuming each value at most twice is called *simple*. We call a compactum *sewable* if it can be mapped by a simple map onto a plane subset having non-empty interior; if not, we call it *non-sewable.*

From widely known general theorems it follows that zero-dimensional compacta and (for other reasons) compacta of dimension greater than 2 are non-sewable. So the problem of sewability of compacta is of interest only when they have dimension 1 or 2. Here we restrict ourselves to the problem of sewability of plane curves, i.e., of plane continua of dimension 1, as we regard this particular case to be sufficiently difficult.

Our first observations suggest that the sewability should be related to the size of the curve: the "small" ones, as arcs and dendrites, are non-sewable, the "big" ones, as the Sierpiński unversal plane curve, are sewable. But this simple correlation fails if we pass to other examples.

A curve is called *regular* if it has a base consisting of sets with finite boundaries. Regular curves are in a certain sense small. However, there exist

(1) Professor Karol Sieklucki has kindly informed us that this result dates back to his unpublished doctoral dissertation.

regular curves which are sewable, even among those of finite order, i.e., having a base consisting of sets cardinalities of whose boundaries have a common bound. In the example which will be given in the final part of these introductory remarks, this bound is 6.

From the Baire theorem it follows that a compactum being the union of countably many non-sewable compacta is non-sewable. Thus the ($\sin 1/x$)-curve is non-sewable, being the union of countably many arcs. Being not locally connected, the ($\sin 1/x$)-curve is not regular.

Thus the non-sewability is not related to the regularity.

The situation described above makes interesting the question: Is the Sierpiński triangular curve sewable? It is regular and is minimal in the sense of order among everywhere ramified curves. Curves having property (α) of Hurewicz, as well as dendrites, are nowhere dense if they are subsets of the Sierpiński triangular curve, so the Baire theorem cannot be applied to deriving its non-sewability from the non-sewability of much more simpler compacta mentioned above. None the less we shall show — and this is the main result of the paper — that the Sierpiński triangular curve is non-sewable.

This result sheds some light on the matter of scope of non-sewable plane curves. However, the whole problem is far from being solved. We do not know, for instance, whether the property of being non-sewable is inherited by subcompacta of the Sierpiński triangular curve. **(P 1383)**

Among the properties of the Sierpiński triangular curve used in the proof of its non-sewability we distinguish the property of preserving interiors under embeddings. This property allows us to reduce the proof to maps called here *conditionally interior preserving*. But the regular sewable curve mentioned above also enjoys this property, and the crucial point of the proof lies in another place, namely in the final part of the proof, where the triangular shape of the curve plays a role, when using Sperner's Pflastersatz we obtain a contradiction in our reasoning a contrario.

The above-mentioned result of Hurewicz [3], even adapted to the particular case considered here, is somewhat stronger than that quoted above. It says that simple maps from curves having property (α) cannot raise the dimension (without assuming that the image lies on the plane); see also Kazhdan [4] and Sieklucki [10]. The same concerns dendrites. However, we do not know whether there exist dimension raising simple maps from the Sierpiński triangular curve. **(P 1384)**

The problem of whether and how the dimension is raised under continuous maps having finite inverse images of points has been extensively treated since Hurewicz's paper in the literature; a rich survey is given by Lelek [7].

An example of a regular sewable curve. Delete from a plane square a lense-shaped domain joining the middle points of the top and the bottom sides of the square. From both the remaining "rectangles" delete two "horizontal" lense-shaped domains joining the middle points of opposite "vertical" sides of these rectangles. Do the analogous "vertical" cuttings in the four "rectangles" just obtained, and so on (see Fig. 1).

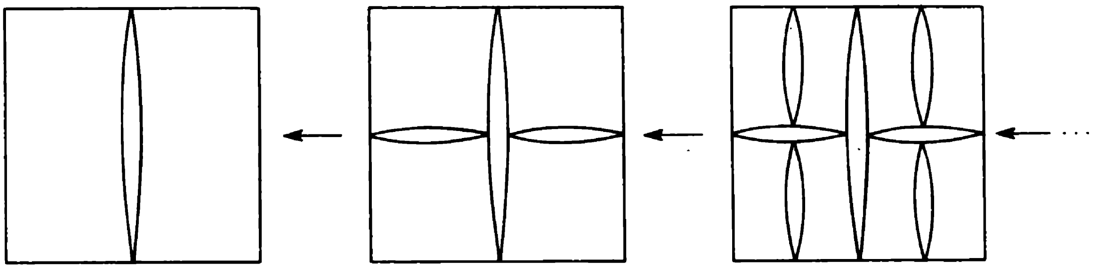

Fig. 1

Sticking together the opposite sides of cuttings, we get from the curve the plane square again. We omit the proof, as the argumentation is the same as that – making use of the Moore decomposition theorem – in [5], where the "sewing" of the Sierpiński plane universal curve was performed.

1. Simple maps from compacta. A space Y will be said to be *of dimension* $\geqslant n$ *everywhere* if the closure of each of its open non-empty subsets is of dimension $\geqslant n$.

In what follows the symbols cl, int and bd stand for closure, interior and boundary of subsets with respect to the entire space under consideration.

LEMMA 1. *Let f be a simple map from a compactum X of dimension $< n$ into a metric space of dimension $\geqslant n$ everywhere. If F is a closed subset of X, then the interiors of $f(F)$ and of $f(X-F)$ are disjoint.*

Proof. If not, there would exist an open non-empty subset V of Y such that

$$\mathrm{cl}\, V \subset \mathrm{int} f(F) \cap \mathrm{int} f(X-F).$$

The map f restricted to $B = F \cap f^{-1}(\mathrm{cl}\, V)$ is then one-to-one, thus a homeomorphism onto cl V, as each its value is assumed also on the set $X-F$ disjoint from B and the map f is simple. But cl V is of dimension $\geqslant n$, the space Y being of dimension $\geqslant n$ everywhere. Thus the subset B of X is of dimension $\geqslant n$, contrary to the assumption that X is of dimension $< n$.

A map $f\colon X \to Y$ will be said to be *conditionally interior preserving* if

$$f(U) \subset \mathrm{int} f(X) \text{ implies } \mathrm{int} f(U) \neq \emptyset$$

for each open non-empty subset U of X.

A subset of a topological space is called *regularly closed* if it is the closure of an open subset, in fact, the closure of its interior.

LEMMA 2. *If $f\colon X \to Y$ is a conditionally interior preserving simple map from a compactum into a metric space and F is a regularly closed subset of X, then*

$$\mathrm{cl}\,\mathrm{int} f(F) \cap \mathrm{int} f(X) = f(F) \cap \mathrm{int} f(X);$$

this means that the sets $f(F)$ and $\mathrm{cl}\,\mathrm{int} f(F)$ *are equal if they are restricted to* $\mathrm{int} f(X)$. *In particular, $f(F)$ is regularly closed if it is contained in* $\mathrm{int} f(X)$.

Proof. To prove the non-obvious inclusion, suppose to the contrary that the set

$$W = (f(F) - \operatorname{cl}\operatorname{int} f(F)) \cap \operatorname{int} f(X)$$

is non-empty. This implies that the set $U = F \cap f^{-1}(W)$, which is open in F, is non-empty. But F is a regularly closed subset of X, and therefore U has non-empty interior in X. We have $f(U) \subset \operatorname{int} f(X)$, as $f(U) \subset W$ and $W \subset \operatorname{int} f(X)$. This implies that $\operatorname{int} f(U) \neq \emptyset$, since f is conditionally interior preserving. Thus $\operatorname{int} W \neq \emptyset$. But, on the other hand, W is nowhere dense, and we get a contradiction.

LEMMA 3. *Let $f\colon X \to Y$ be a conditionally interior preserving simple map from a compactum of dimension $< n$ into a metric space of dimension $\geqslant n$ everywhere. If F is a closed subset of X, then*

$$f(\operatorname{bd} F) \subset \operatorname{bd} f(F).$$

Proof. Suppose to the contrary that the inclusion does not hold, i.e., that

$$f(\operatorname{bd} F) \cap \operatorname{int} f(F) \neq \emptyset.$$

Hence $\operatorname{bd} F \cap f^{-1}(\operatorname{int} f(F)) \neq \emptyset$. Thus the open set

$$U = f^{-1}(\operatorname{int} f(F)) - F$$

is non-empty.

We have

$$f(U) \subset \operatorname{int} f(F) \subset \operatorname{int} f(X),$$

which implies that $\operatorname{int} f(U) \neq \emptyset$, as f is conditionally interior preserving.

On the other hand, $f(U) \subset f(X-F)$, and we see that the open and non-empty set $\operatorname{int} f(U)$ is contained in the intersection of $f(F)$ and $f(X-F)$, contrary to Lemma 1.

COROLLARY. *Let $f\colon X \to Y$ be a conditionally interior preserving simple map from a compactum of dimension $< n$ into a metric space of dimension $\geqslant n$ everywhere. If F is a closed subset of X such that $f(F) \subset \operatorname{int} f(X)$, then*

$$\operatorname{int} f(F) = f(X) - f(X - \operatorname{int} F).$$

Proof. In the proof of the inclusion

$$f(X) - f(X - \operatorname{int} F) \subset \operatorname{int} f(F)$$

none of the preceding lemmas is needed. Observe namely that the set on the left-hand side is contained in $f(F)$, since from the obvious equality

$$(X - \operatorname{int} F) \cup F = X$$

it follows that

$$f(X - \operatorname{int} F) \cup f(F) = f(X).$$

Thus, being open in $f(X)$, it is contained in the interior of $f(F)$ relatively to $f(X)$. But, by assumption, $f(F)$ is contained in the interior of $f(X)$. Hence the relative interior of $f(F)$ with respect to $f(X)$ equals its interior in the entire space Y, i.e., $\operatorname{int} f(F)$. Thus the desired inclusion holds.

To show the converse inclusion observe that

$$\operatorname{int} f(X-F) \cap \operatorname{int} f(F) = \emptyset$$

by Lemma 1. By Lemma 3, this is strengthened to the disjointness of the interiors of the sets $A = f(X - \operatorname{int} F)$ and $f(F)$.

Clearly, we have also $\operatorname{cl}\operatorname{int} A \cap \operatorname{int} f(F) = \emptyset$. Applying Lemma 2 to $X - \operatorname{int} F$ in place of F, we get

$$\operatorname{cl}\operatorname{int} A \cap \operatorname{int} f(X) = A \cap \operatorname{int} f(X),$$

which leads to

$$\operatorname{cl}\operatorname{int} A \cap \operatorname{int} f(F) = A \cap \operatorname{int} f(F)$$

and, in consequence, to $A \cap \operatorname{int} f(F) = \emptyset$, i.e., to the equality

$$f(X - \operatorname{int} F) \cap \operatorname{int} f(F) = \emptyset,$$

which implies the needed inclusion.

A space is said to have the *property of preserving interiors* (*under embeddings*) if homeomorphic copies of its open non-empty subsets contained in this space have non-empty interiors.

THEOREM 1. *If f is a simple map from a compactum having the property of preserving interiors into a metric space, then f is conditionally interior preserving.*

Proof. Let X be a compactum having the property of preserving interiors and let f be a simple map from X into a metric space Y.

Let U be a non-empty open subset of X such that $f(U) \subset \operatorname{int} f(X)$. Suppose that $\operatorname{int} f(U) = \emptyset$. We have $f(X-U) = f(X)$. Let V be an open non-empty subset of X such that $\operatorname{cl} V \subset U$. Each value assumed by f on $\operatorname{cl} V$ is assumed also on $X - U$. Thus the inverse image $f^{-1}(f(\operatorname{cl} V))$ of $f(\operatorname{cl} V)$ splits into two disjoint compact sets, one of which is $\operatorname{cl} V$ and the other

$$A = (X-U) \cap f^{-1}(f(\operatorname{cl} V)),$$

lying in $X-U$. Since f is simple, its restrictions to each of these sets are one-to-one and, therefore, are homeomorphisms onto $f(\operatorname{cl} V)$.

The inverse image $f^{-1}(f(V))$ of $f(V)$ splits also into two disjoint sets, one of which is V and the other, call it B, lying in A. Both these sets are mapped by f onto $f(V)$ homeomorphically, as $f|V$ and $f|B$ are restrictions of homeomorphisms. It follows that V and B are homeomorphic. But X has the property of preserving interiors. Thus $\operatorname{int} B \neq \emptyset$.

Now, note that the sets $f(X-(V \cup \operatorname{int} B))$ and $f(\operatorname{int} B)$ cover $f(X)$, as $f(X-V) = f(X)$. Moreover, these sets are disjoint, since $f(\operatorname{int} B) \subset f(V)$ and f is simple. The set $f(\operatorname{int} B)$ is open in $f(X)$, being the complement of the closed subset $f(X-(V\cup \operatorname{int} B))$ of $f(X)$. But

$$f(\operatorname{int} B)\subset f(V)\subset \operatorname{int} f(X),$$

and therefore $f(\operatorname{int} B)$ is open in Y. Thus $\operatorname{int} f(V)\neq \emptyset$, the set $\operatorname{int} B$ being non-empty. We get a contradiction with $\operatorname{int} f(U)=\emptyset$.

2. Conditionally interior preserving simple maps into E^n which raise dimension. We shall consider here conditionally interior preserving simple maps from a compactum X of dimension $< n$ into E^n, the Euclidean n-dimensional space. As E^n is of dimension $\geqslant n$ everywhere, the lemmas from the preceding section are applicable.

For a given conditionally interior preserving simple map $f\colon X\to E^n$, where $n\geqslant 2$, we shall prove some properties of images of closed subsets F of X.

In the first of the announced lemmas the assumption that f is conditionally interior preserving is not required.

LEMMA 4. *If a regularly closed subset F of X has connected interior and this interior is not disconnected by points, then $f(F)$ is connected and is not disconnected by points belonging to* $\operatorname{int} f(X)$.

Proof. The connectedness of $f(F)$ is obvious.

Suppose that p disconnects $f(F)$. This implies that the set $f^{-1}(p)=\{a, b\}$ disconnects F. We have $a\neq b$, since F, being the closure of $\operatorname{int} F$, is connected and not disconnected by points; $\operatorname{int} F$ has these properties by assumption. The points a and b lie in the interior of F; otherwise, one of them would disconnect $\operatorname{int} F$.

Suppose that $p\in \operatorname{int} f(X)$. Now, in each neighbourhood of p there are values of f assumed on $X-F$, since $f(F)$, being disconnected by p, cannot fill any whole neighbourhood of p in E^n if $n\geqslant 2$; in particular, any such neighbourhood which is contained in $f(X)$. By compactness and continuity, p would be the value of f assumed at a point of $X-\operatorname{int} F$ (the set which is the closure of $X-F$), i.e., at some point other than a and b. A contradiction, for f is simple.

LEMMA 5. *If F is a closed subset of X such that $f(F)\subset \operatorname{int} f(X)$ and such that the complement $X-F$ of F is connected and not disconnected by points, then the complement $E^n-\operatorname{int} f(F)$ of $f(F)$ is connected and not disconnected by points.*

Proof. We have

$$E^n-\operatorname{int} f(F)=f(X-\operatorname{int} F)\cup (E^n-f(X)).$$

Indeed, from the Corollary to Lemmas 1–3 it follows that

$$f(X)-f(X-\operatorname{int} F)=\operatorname{int} f(F)$$

and, in consequence, we have the desired equality.

The set $f(X-\operatorname{int} F)$ is connected, since $X-\operatorname{int} F$ is connected and is not disconnected by points of $\operatorname{int} f(X)$; the latter is a consequence of Lemma 4 applied to $X-\operatorname{int} F$ in place of F.

The set $E^n - f(X)$ is the union of regions of E^n whose non-degenerate boundaries lie on the boundary of $f(X)$. But the boundary of $f(X)$ is contained in $f(X - \operatorname{int} F)$, as $f(F) \subset \operatorname{int} f(X)$. Thus the union of $E^n - f(X)$ and $f(X - \operatorname{int} F)$, i.e., the set $E^n - \operatorname{int} f(F)$, is connected. It is not disconnected by points. Indeed, it is obviously not disconnected by points of the open set $E^n - f(F)$. The points of $f(F)$, as we have seen, do not disconnect $f(X - \operatorname{int} F)$ and they do not lie on the closure of $E^n - f(X)$; therefore, they also do not disconnect $E^n - \operatorname{int} f(F)$.

LEMMA 6. *Let F be a regularly closed subset of X such that* $\operatorname{int} F$ *and $X - F$ are connected and not disconnected by points, and such that $f(F) \subset \operatorname{int} f(X)$. The boundary* $\operatorname{bd} f(F)$ *of $f(F)$ is connected and not disconnected by points.*

Proof. We have

$$\operatorname{bd} f(F) = f(F) \cap (E^n - \operatorname{int} f(F)),$$

since $f(F)$ is closed.

We have obviously

$$f(F) \cup (E^n - \operatorname{int} f(F)) = E^n.$$

Thus, from the unicoherence of E^n (see, e.g., [12], p. 228, Corollary 7.31) we infer that the intersection $\operatorname{bd} f(F)$ of $f(F)$ and $E^n - \operatorname{int} f(F)$ is connected, both these sets being connected.

Remove a point p from $\operatorname{bd} f(F)$. We have

$$\operatorname{bd} f(F) - \{p\} = (f(F) - \{p\}) \cap \big(E^n - (\operatorname{int} f(F) \cup \{p\})\big).$$

Add the point at infinity to E^n obtaining the sphere S^n. We have

$$S^n - \{p\} = (f(F) - \{p\}) \cup \big(E^n - (\operatorname{int} f(F) \cup \{p\})\big),$$

i.e., again a decomposition of the Euclidean n-dimensional space $S^n - \{p\}$ into two connected sets, since $f(F)$ and $E^n - \operatorname{int} f(F)$ are, by Lemmas 4 and 5, connected and not disconnected by points. As before, from the unicoherence of $S^n - \{p\}$, $n \geqslant 2$, we infer that the intersection $\operatorname{bd} f(F) - \{p\}$ of these sets is connected. This completes the proof.

If a compact subset A of E^n is connected and not disconnected by points, then the boundaries of complementary regions of A are connected and not disconnected by points. Applying this theorem known in the topology of Euclidean spaces (see, e.g., [6], Section 53, IV.4, p. 349), we get from Lemma 6 (assuming that F satisfies the hypotheses of that lemma) the following

COROLLARY. *Let U be a complementary region of* $\operatorname{bd} f(F)$. *The boundary* $\operatorname{bd} U$ *of U is connected and not disconnected by points.*

3. The Sierpiński triangular curve. Let T be an equilateral triangle on the plane. Divide T into four congruent triangles and remove the interior of the middle one. The remaining triangles T_0, T_1 and T_2 have diameters equal to the half of that of T and the union $T^{(1)}$ of them is connected (see Fig. 2).

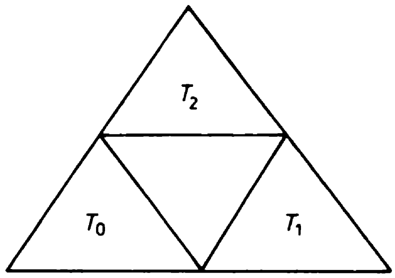

Fig. 2

Apply the same procedure of dividing to the triangles T_i, and then to the triangles obtained recursively in this way. We get at the n-th stage of the procedure 3^n congruent triangles of diameters equal to the $(1/2^n)$-th of the diameter of T. The union $T^{(n)}$ of the triangles obtained at the n-th stage is connected, thus a plane continuum. The intersection

$$S = T^{(1)} \cap T^{(2)} \cap \ldots$$

is the *Sierpiński triangular curve*. It was described by Sierpiński in [11].

Call the intersections with S of triangles obtained in this procedure the *triangles of S*.

The triangles $S \cap T_i$ will be called the *main subtriangles* of S. The common points between adjacent triangles of S will be called *links*; they are common *vertices* of triangles.

The following topological properties of S are obvious:

(A) S is not disconnected by points.

(B) A triple disconnecting S into three connected sets is the triple of links between main subtriangles of S.

(C) No triple disconnects S into more than three connected sets.

THEOREM 2. *A subspace of S homeomorphic to S is a triangle of S.*

Going into details: a topological embedding of S into S is a homothety onto a triangle of S and this homothety is determined by the position under the embedding of the triple of links between main subtriangles of S.

Proof. Let h be an embedding of S into S. Let $X = h(S)$. Let Δ be the smallest triangle of S containing X. Let Δ_0, Δ_1 and Δ_2 be main subtriangles of Δ. The set X cannot lie in the union of any two of them, since in that case it would be disconnected by the common vertex of these subtriangles, contrary to the fact that X, being homeomorphic to S, cannot be disconnected by points (property (A)).

This means that X has points in geometric interiors of all subtriangles Δ_i of Δ, i.e., intersects these subtriangles not only at their vertices.

All three links between Δ_i belong to X, since in the case where one of them misses X, each of the remaining two would disconnect X, again contrary to property (A).

Thus the triple consisting of links between Δ_i disconnects the set X into at least three connected sets and, in consequence, by property (C), into exactly three connected sets. In view of property (B), this triple is the image of the triple of links between subtriangles $S \cap T_i$ of S.

Now, we shall show that

(*) *the main subtriangles $S \cap T_i$ of S are mapped by h into the corresponding main subtriangles of Δ.*

To see this, let a triangle $S \cap T_i$ be fixed and let x be the link between the remaining two main subtriangles of S. The value $h(x)$ at x is, according to the remark made above, a link between some of two main subtriangles of Δ. Denote by Δ_i the remaining subtriangle (the opposite to the link just mentioned). The image of $S \cap T_i$ under h is contained in the subtriangle Δ_i indicated above. Indeed, the image of $S \cap T_i$ does not contain $h(x)$. Therefore, it must be contained in Δ_i, since otherwise it would be disconnected by one of the remaining links. This is impossible since $S \cap T_i$, being homeomorphic to S, cannot be disconnected by single points (property (A) of S).

Now, we shall apply the preceding reasoning to the partial embeddings of $S \cap T_j$ into Δ_j. This can be done, since from (*) it follows that the image of $S \cap T_j$ under h does not lie in a single main subtriangle of Δ_j.

As before, the triple consisting of links between main subtriangles of $S \cap T_j$ is mapped under h onto the triple consisting of links between main subtriangles of Δ_j. But now, the values of h at these links are determined by the values of h on the triple of links between main subtriangles of S which are known from the preceding step.

The process continues and we see that S embeds under h densely into Δ and that the embedding is a homothety on the set of links of S, this homothety being determined by the values of h on the triple of main links of S. Thus, by compactness and continuity, the embedding h is the homothety between S and Δ determined by the initial condition mentioned above.

From Theorem 2 the following immediately results:

COROLLARY. *If U is an open non-empty subset of S and V is a homeomorphic copy of U contained in S, then the interior of V is non-empty*; *in other words, the Sierpiński triangular curve has the property of preserving interiors* (*under embeddings*).

In fact, we can say more: if U does not contain vertices of the triangle T, then the set V is open. Thus the Sierpiński triangular curve has the property of preserving openness in the sense commonly used if we neglect the above-mentioned singularity.

From the last Corollary and Theorem 1 we get immediately

COROLLARY. *Simple maps from the Sierpiński triangular curve into the plane are conditionally interior preserving.*

Note. It is not difficult to repeat the reasoning from the proof of Theorem 2 and to show that the "rectangular" curve described in the introductory section enjoys a property analogous to that proved for the Sierpiński triangular curve, in which the role of subtriangles will be played by "subrectangles". We do not insert the proof because this fact will be not used in our deduction, serving only as a support for the comment we have made in the introduction.

Let us note the following easy property of the Sierpiński triangular curve S:

(D) The complement $S-\triangle$ of any triangle $\triangle$ of S is connected; if, in addition, the triangle $\triangle$ lies in the plane interior of T, then $S-\triangle$ is not disconnected by points and all three vertices of $\triangle$ lie on the topological boundary $\mathrm{bd}\,\triangle$ of $\triangle$ with respect to S.

4. Main theorem. We know from Section 3 that simple maps from the Sierpiński triangular curve into the plane are conditionally interior preserving. Some properties of images of closed and regularly closed subsets under conditionally interior preserving simple maps from compacta into E^n, in particular into the plane, were studied in Section 2. This preparatory material will be used now in the proof of the following

MAIN THEOREM. *There does not exist a simple map from the Sierpiński triangular curve onto a plane subset having non-empty interior.*

In other words: the Sierpiński triangular curve is non-sewable.

Proof. Assume to the contrary that we have a simple map f from the Sierpiński triangular curve S into the plane such that

$$\mathrm{int} f(S) \neq \emptyset.$$

Thus there exist triangles $\triangle$ of S (also among those which lie in the plane interior of T) such that

(i) $f(\triangle) \subset \mathrm{int} f(S)$.

According to property (D) of S, if $\triangle$ is such a triangle, then

(ii) $S-\triangle$ is connected and not disconnected by points and all three vertices of $\triangle$ lie on the topological boundary $\mathrm{bd}\,\triangle$ of $\triangle$ with respect to S.

The map f, being a simple map from the Sierpiński triangular curve S, is conditionally interior preserving. The dimension of S is 1 and the values of f lie in E^2, so the preparatory material from Section 2 concerning images of closed subsets F of compacta considered there can be applied to triangles $\triangle$ of S in place of F. Note that triangles of S are regularly closed subsets of S.

Let $\triangle$ be a triangle of S having properties (i) and (ii). Let a, b and c be the vertices of $\triangle$. According to (ii), the vertices of $\triangle$ lie on $\mathrm{bd}\,\triangle$. Since (i) holds, Lemma 3 can be applied. We infer that the values $f(a)$, $f(b)$ and $f(c)$ lie on $\mathrm{bd} f(\triangle)$.

Points of $\mathrm{bd} f(\triangle)$ different from $f(a)$, $f(b)$ and $f(c)$ are single values of $f|\triangle$ since these points are also values at some points of $S-\triangle$; this follows easily from the fact that $f(\triangle) \subset \mathrm{int} f(S)$.

The values $f(a)$, $f(b)$ and $f(c)$ may or may not be single values of f.

A triangle $\triangle$ of S will be said to be *of type* $[1, *, *]$ if at least one value, $f(a)$,

$f(b)$ or $f(c)$, is single for $f|\Delta$; similarly, a triangle Δ will be said to be *of type* $[1, 1, *]$ or *of type* $[1, 1, 1]$ if at least two or (in the latter case) all values $f(a)$, $f(b)$ and $f(c)$ are single for $f|\Delta$.

Observe that for each triangle Δ of S at least one of its main subtriangles is of type $[1, *, *]$.

Indeed, let us denote by x, y and z the links between the main subtriangles of Δ, as in Fig. 3. If the value $f(x)$ at the link x is double for f restricted to the subtriangle at b, it is single for the subtriangle at c.

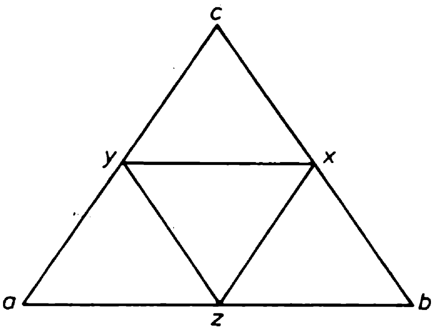

Fig. 3

Now, observe that for each triangle Δ of type $[1, *, *]$ at least one of its main subtriangles is of type $[1, 1, *]$.

To see this, let $f(a)$ be a single value of $f|\Delta$. As before, the values at the links x, y and z are single for at least one of the adjacent main subtriangles of Δ. If $f(y)$ or $f(z)$ is a single value for the subtriangle at a, then this subtriangle is of type $[1, 1, *]$. If not, the values $f(y)$ and $f(z)$ are single for the adjacent subtriangles (at b and at c), and the value $f(x)$ is single for one of these subtriangles. Thus, one of them is of type $[1, 1, *]$.

Now, we shall show that if Δ is a triangle of type $[1, 1, *]$, then at least one of its main subtriangles is of type $[1, 1, 1]$; moreover, we shall show that if Δ is of type $[1, 1, 1]$, then all of main subtriangles of Δ (and therefore all the subtriangles of Δ) are of type $[1, 1, 1]$.

1. To prove the first part of the assertion assume that $f(a)$ and $f(b)$ are single values for $f|\Delta$ and that the value $f(c)$ is double.

Clearly, $f(a) \neq f(b)$ and $f(a) \neq f(c) \neq f(b)$. In the case considered here, according to a remark at the beginning of the proof of the theorem, all the points from $\mathrm{bd} f(\Delta)$ are single values of $f|\Delta$, except the value $f(c)$, which is double.

By Lemma 6, $\mathrm{bd} f(\Delta)$ is connected, so the inverse image $\Delta \cap f^{-1}(\mathrm{bd} f(\Delta))$ consists of at most two connected components, since only one double value appears on $\mathrm{bd} f(\Delta)$, other being single. Only one of these components can be non-degenerate, since in the other case the images of both, being non-degenerate continua (f cannot lower dimension, being simple) intersecting at a single point, would have as their union a continuum disconnected by a point (point $f(c)$). On the other hand, this union is the whole $\mathrm{bd} f(\Delta)$ and, by Lemma 6, it is a continuum not disconnected by points.

Denote by C the non-degenerate component of $\Delta \cap f^{-1}(\mathrm{bd} f(\Delta))$. Since the other is degenerate, we have

$$f(C) = \mathrm{bd} f(\Delta).$$

We have $a \in C$ and $b \in C$, since the value at the single point of the degenerate component must be $f(c)$.

Consider the possible positions of C in Δ and in its main subtriangles, preserving the notation from Fig. 3.

The case where the links x and y belong to C.

The value $f(x)$ is then a single value for the subtriangle at b (a double value of $f|\Delta$ must be the value at c, at a point not belonging to the subtriangle at b). For the same reasons the value $f(y)$ is a single value for the subtriangle at a. The value $f(z)$ is single for one of the subtriangles having a or b as a vertex. One of these subtriangles is therefore of type $[1, 1, 1]$.

The case where x or y does not lie on C.

Now, $z \in C$ since C joins a and b in Δ.

In the subcase where $x \in C$, the subtriangle at b is of type $[1, 1, 1]$.

Similarly, if $y \in C$, the subtriangle at a is of type $[1, 1, 1]$.

Consider the remaining subcase where $x \notin C$ and $y \notin C$.

Now, C is contained in two subtriangles, those at a and at b, and therefore z disconnects C. We have also $c \notin C$, so $f|C$ is a homeomorphism onto $\mathrm{bd} f(\Delta)$. Thus we infer that $\mathrm{bd} f(\Delta)$ is disconnected by $f(z)$, again contrary to Lemma 6. So the subcase where $x \notin C$ and $y \notin C$ is contradictory.

2. Assume now that all three values $f(a)$, $f(b)$ and $f(c)$ are single for $f|\Delta$. Now, all the values of $f|\Delta$ lying on $\mathrm{bd} f(\Delta)$ are single, and C, defined as before as the non-degenerate component of $\Delta \cap f^{-1}(\mathrm{bd} f(\Delta))$, is a continuum homeomorphic to $\mathrm{bd} f(\Delta)$ under the map $f|C$. The continuum C must contain all three links x, y and z, since in the other case C, and in consequence $\mathrm{bd} f(\Delta)$, would be disconnected by points, and this is impossible by Lemma 6. The values $f(x)$, $f(y)$ and $f(z)$, lying on $f(C) = \mathrm{bd} f(\Delta)$, are single for $f|\Delta$. Clearly, they are single for f restricted to subtriangles of Δ. Thus all subtriangles of Δ are of type $[1, 1, 1]$.

From the above considerations it follows that there exist triangles Δ of S satisfying (i), lying in the plane interior of T, and therefore satisfying (ii) and all the conditions assumed in lemmas in Section 1 concerning F and being of type $[1, 1, 1]$. Observe that all these properties are inherited by all subtriangles of such a triangle Δ.

The proof will be completed if we show that

(**) *the image under f of a triangle Δ of type $[1, 1, 1]$ lying in the plane interior of T is a nowhere dense subset of the plane.*

The proof of (**) will consist in showing that

(***) $f(\Delta) \subset \mathrm{bd} f(\Delta)$.

Proof of (***). Let $C = \Delta \cap f^{-1}(\mathrm{bd} f(\Delta))$. The triangle Δ is of type $[1, 1, 1]$, so C is a continuum, $f|C$ being a homeomorphism onto $\mathrm{bd} f(\Delta)$. The continuum C must contain, besides the vertices a, b and c, the links x, y and z, since otherwise it would be disconnected by points, contrary to Lemma 6. Thus $f(x), f(y)$ and $f(z)$ belong to $\mathrm{bd} f(\Delta)$.

Let $C_i = C \cap \Delta_i$, where Δ_i stand for main subtriangles of Δ.

Observe that C_i are connected. Otherwise, the continuum C would be disconnected by a single point, namely by a link between some of two main subtriangles of Δ. But C is homeomorphic to $\mathrm{bd} f(\Delta)$, and we get a contradiction with Lemma 6.

The continua $C_i' = f(C_i)$ form a collection of three plane continua each two of which intersect at a single point, distinct for each pair. These points are the values of f at the links x, y and z.

We shall show that

(1) $$\mathrm{int} f(\Delta) \subset \mathrm{int} f(\Delta_0) \cup \mathrm{int} f(\Delta_1) \cup \mathrm{int} f(\Delta_2).$$

To show this, let $p \in \mathrm{int} f(\Delta)$. Let U be that connected component of $\mathrm{int} f(\Delta)$ to which p belongs. It is, at the same time, a connected component of $E^2 - \mathrm{bd} f(\Delta)$. Thus, from the Corollary to Lemma 6 it follows that the boundary of U is connected and not disconnected by points. But $\mathrm{bd} f(\Delta)$ is hereditarily locally connected, being homeomorphic to a subcontinuum of the Sierpiński triangular curve. Thus $\mathrm{bd}\, U$ is locally connected, being contained in $\mathrm{bd} f(\Delta)$ (recall that, by the Corollary to Lemma 3, the set $f(\Delta)$ is regularly closed, and therefore equal to the closure of $\mathrm{int} f(\Delta)$).

A locally connected boundary of a plane region, if it is not disconnected by points, is homeomorphic to the circle (see, e.g., [6], Chapter IX, Section 54, II, 4 (ii), p. 360). Thus the boundary $\mathrm{bd}\, U$ of U is homeomorphic to the circle.

By the Schönflies theorem (see also [6], Chapter IX, 54, V,2, Corollary, p. 381), the closure $\mathrm{cl} U$ of U is homeomorphic to the closed disc.

We shall show that

(2) $$\mathrm{bd}\, U \subset C_i' \quad \text{for some } i.$$

Clearly, $\mathrm{bd}\, U$ is contained in the union of C_i', since this union is contained in $\mathrm{bd} f(\Delta)$.

If $\mathrm{bd}\, U$ is not contained in one of C_i', it cannot be contained in the union of two of them, since the single common point of these sets would disconnect $\mathrm{bd}\, U$, but this is impossible as $\mathrm{bd}\, U$ is a circle.

Thus, to show (2), it remains to exclude the case where $\mathrm{bd}\, U$ intersects each C_i' and is contained in no two of them. In this case, all three points $f(x), f(y)$ and $f(z)$ lie on the circle $\mathrm{bd}\, U$, and they divide this circle into three arcs

$$L_0 = [f(y), f(z)], \quad L_1 = [f(z), f(x)] \quad \text{and} \quad L_2 = [f(x), f(y)],$$

having only ends in common.

The arcs L_i are contained in the union of the continua C'_j. Since each two of these continua intersect at a single point from the triple $f(x), f(y), f(z)$, different for each couple, each arc L_i is contained in one of C'_j.

We have $L_0 \subset C'_0$, since $f(y)$ and $f(z)$ belong to both L_0 and C'_0. Similarly, $L_1 \subset C'_1$ and $L_2 \subset C'_2$.

Thus,

$$L_i = C'_i \cap \operatorname{bd} U. \tag{3}$$

Observe that

$$f(\Delta_i) \cap \operatorname{bd} U = L_i. \tag{4}$$

Indeed, all the points of $\operatorname{bd} f(\Delta)$ are single values for $f|\Delta$, since the triangle Δ is of type $[1, 1, 1]$. Thus we have

$$f(\Delta_i) \cap \operatorname{bd} f(\Delta) = f(\Delta_i) \cap f(C) = f(\Delta_i \cap C) = f(C_i) = C'_i$$

and, in consequence,

$$f(\Delta_i) \cap \operatorname{bd} U = f(\Delta_i) \cap \operatorname{bd} f(\Delta) \cap \operatorname{bd} U = C'_i \cap \operatorname{bd} U = L_i,$$

the last equality being derived from (3), and the first one from the inclusion $\operatorname{bd} U \subset \operatorname{bd} f(\Delta)$.

The sets $f(\Delta_i)$ cover the disc $\operatorname{cl} U$. They dissect (see (4)) from the boundary of the disc the arcs L_i each two of them having only an end in common. Thus Sperner's Pflastersatz can be applied, and we conclude that there exists a point common for all the sets $f(\Delta_i) \cap \operatorname{cl} U$. This point lies in U since

$$\operatorname{bd} U \cap f(\Delta_i) = L_i \quad \text{and} \quad L_0 \cap L_1 \cap L_2 = \emptyset.$$

This common point is the value of f at points of Δ_0, Δ_1 and Δ_2, and none of these points is any link x, y and z, since $f(x), f(y)$ and $f(z)$ lie on $\operatorname{bd} U$. Thus we get a point being the value of f at three points of S. This contradiction completes the proof of (2).

Now, observe that

$$\operatorname{bd} U \subset C'_i \ \text{ implies } \ U \subset f(\Delta_i). \tag{5}$$

Since $U \subset f(\Delta_0) \cup f(\Delta_1) \cup f(\Delta_2)$, it suffices to show that U is disjoint from the set $f(\Delta_j \cup \Delta_k - \Delta_i)$, i, j and k being different.

The set mentioned above is connected, being the image of the connected set $\Delta_j \cup \Delta_k - \Delta_i$. It is disjoint from $C'_i = f(\Delta_i \cap C)$, since the values assumed on C are single for f and the sets $\Delta_i \cap C$ and $\Delta_j \cap \Delta_k - \Delta_i$ are disjoint.

Thus, the set $f(\Delta_j \cap \Delta_k - \Delta_i)$ is disjoint from $\operatorname{bd} U$ since, by assumption, $\operatorname{bd} U \subset C'_i$. It lies in one of the complementary regions of $\operatorname{bd} U$, and this region is different from U, since it has points outside of U, for instance the point $f(t)$, where t is the link between Δ_j and Δ_k. The point $f(t)$ lies on $\operatorname{bd} f(\Delta)$ and U does not intersect $\operatorname{bd} f(\Delta)$, being one of the complementary regions of $\operatorname{bd} f(\Delta)$. Thus (5) is proved.

From (4) and (5) it follows that $U \subset f(\Delta_i)$ for some i. Thus $p \in \operatorname{int} f(\Delta_i)$ for this i, U being open. Since p is an arbitrary point of $\operatorname{int} f(\Delta)$, we have shown that

$$\operatorname{int} f(\Delta) \subset \operatorname{int} f(\Delta_0) \cup \operatorname{int} f(\Delta_1) \cup \operatorname{int} f(\Delta_2).$$

This implies that

$$\operatorname{bd} f(\Delta_i) \subset \operatorname{bd} f(\Delta),$$

since in the other case $\operatorname{int} f(\Delta_i) \cap \operatorname{int} f(\Delta_j) \neq \emptyset$ for some $j, j \neq i$; by Lemma 1 of Section 1, this is impossible.

We can iterate the procedure. We get

$$\operatorname{bd} f(\Delta') \subset \operatorname{bd} f(\Delta)$$

and, in particular,

$$f(\Delta') \cap \operatorname{bd} f(\Delta) \neq \emptyset$$

for the subtriangles Δ' of Δ of arbitrary range.

Let $\varepsilon > 0$ be given. Let n be such that the images of triangles Δ' of range n have diameters not greater than ε. Let q be a point of $f(\Delta)$. Let Δ' be that of subtriangles of Δ of range n for which $q \in f(\Delta')$. From $f(\Delta') \cap \operatorname{bd} f(\Delta) \neq \emptyset$ it follows that the distance from q to $\operatorname{bd} f(\Delta)$ is not greater than ε. But q has been taken in $f(\Delta)$ arbitrarily. Also a positive ε is arbitrary. Thus we infer that $f(\Delta)$ lies in the closure of $\operatorname{bd} f(\Delta)$, i.e., in $\operatorname{bd} f(\Delta)$.

Thus we have proved (***), and the proof of our theorem is completed.

Acknowledgement. The authors would like to thank the referee for his attention in reading the paper, which allowed us to avoid some small but unpleasant mistakes.

REFERENCES

[1] K. Borsuk and R. Molski, *On a class of continuous mappings*, Fund. Math. 45 (1958), pp. 84–88.

[2] H. Hahn, *Über die Abbildung einer Strecke auf ein Quadrat*, Annali di Matematica Serie III, 21 (1913), pp. 33–35.

[3] W. Hurewicz, *Über dimensionserhöhende stetige Abbildungen*, J. Reine Angew. Math. 169 (1933), pp. 99–106.

[4] Ya. M. Kazhdan, *On continuous mappings which increase dimension* (in Russian), Dokl. Akad. Nauk SSSR 67 (1949), pp. 19–22.

[5] B. Knaster and A. Lelek, *Coutures et tapis*, Fund. Math. 45 (1958), pp. 186–199.

[6] C. Kuratowski, *Topologie, II*, Monograf. Mat. 21, PWN, Warszawa–Wrocław 1950.

[7] A. Lelek, *On Peano mappings* (in Polish), Prace Mat. 7 (1962), pp. 127–140.

[8] — *Dimension inequalities for unions and mappings of separable metric spaces*, Colloq. Math. 23 (1971), pp. 69–91.

[9] S. Mazurkiewicz, *Sur les points multiples des courbes qui remplissent une aire plane* (in Polish), Prace Mat.-Fiz. 26 (1915), pp. 113–120; for the French translation, see pp. 48–53 in: *Travaux de topologie* by S. Mazurkiewicz, PWN, Warszawa 1969.

[10] K. Sieklucki, *A generalization of a theorem of Mazurkiewicz concerning Peano functions*, Prace Mat. 12 (1969), pp. 251–253.

[11] W. Sierpiński, *Sur une courbe dont tout point est un point de ramification* (in Polish), Prace Mat.-Fiz. 27 (1916), pp. 77–86; for the French translation, see pp. 99–106 in: Vol. II of *Oeuvres choisies* by W. Sierpiński, PWN, Warszawa 1977.

[12] G. T. Whyburn, *Analytic Topology*, Amer. Math. Soc. Colloq. Publ. 28, Providence 1942.

INSTYTUT MATEMATYKI
UNIWERSYTET ŚLĄSKI
KATOWICE

Reçu par la Rédaction le 29.6.1988;
en version modifiée le 25.1.1989